Lagrange Multipliers - Extremizing the volume of a box

Problem : Given a closed rectangular box, of surface area S and total edge length L, extremize the volume of the box.

Let the lengths of the edges be x,y and z.
Then, the problem is to extremize xyz subject to the conditions

$$\begin{aligned} 2xy + 2yz + 2zx = S \\ 4x + 4y + 4z = L \end{aligned} \tag{1}$$

We set up the *Lagrangian*

$$F(x, y, z, \lambda, \mu) = xyz - \lambda(2xy + 2yz + 2zx - S) - \mu(2x + 2y + 2z - L) \tag{2}$$

which gives us the set of equations

$$\begin{aligned} yz - \lambda(2y + 2z) - 4\mu = 0 \\ xz - \lambda(2x + 2z) - 4\mu = 0 \\ xy - \lambda(2x + 2y) - 4\mu = 0 \\ 2xy + 2yz + 2zx = S \\ 4x + 4y + 4z = L \end{aligned} \tag{3}$$

Subtracting the first 3 equations pairwise and factoring yields

$$\begin{aligned} (y - x)(z - 2\lambda) = 0 \\ (z - y)(x - 2\lambda) = 0 \\ (x - z)(y - 2\lambda) = 0 \end{aligned} \tag{4}$$

Then $x = y$ or $z = 2\lambda$. In the second case, the other two equations then become

$$\begin{aligned} (2\lambda - y)(x - 2\lambda) = 0 \\ (x - 2\lambda)(y - 2\lambda) = 0 \end{aligned} \tag{5}$$

which gives $x = z$ or $y = z$, and is equivalent to $x = y$.

We thus may assume that $x = y$, which gives the two cases :

Case 1 : $x = y = z$ (a cube). This is only possible if $L^2 = 24S$, and gives only one possible solution $x = y = z = \dfrac{L}{12}$, $V = xyz = \dfrac{L^3}{1728}$.

Case 2 : $x = y = 2\lambda$. Solving for z in the equation for the edge length yields $z = \dfrac{L - 16\lambda}{4}$.
We substitute these values into the equation for the surface area and obtain after simplification

$$24\lambda^2 - 2L\lambda + S = 0 \tag{6}$$

Examining the discriminant of this quadratic shows that we only have a solution if

$$L^2 - 24S \geq 0 \tag{7}$$

Putting all of these back into the original equations yields 2 distinct solutions

$$\begin{aligned}
&D = \sqrt{L^2 - 24S} \\
&x = \tfrac{1}{12}(L \pm D) \\
&y = \tfrac{1}{12}(L \pm D) \\
&z = \tfrac{1}{12}(L \mp 2D) \\
&\lambda = \tfrac{1}{24}(L \pm D) \\
&\mu = -\lambda^2 = -\tfrac{1}{576}\left(2L^2 - 24S \pm 2LD\right)
\end{aligned} \tag{8}$$

The extreme volumes are then given by

$$\begin{aligned}
&V_{min} = \tfrac{1}{1728}\left(24LS - 2D^3\right) \\
&V_{max} = \tfrac{1}{1728}\left(24LS + 2D^3\right)
\end{aligned} \tag{9}$$

In the special case when $S = 1500\ cm^3$, $L = 200\ cm$, we obtain (to 3 decimal places)

$$\begin{aligned}
&x = 21.937\ cm \\
&y = 21.937\ cm \\
&z = 6.126\ cm \\
&\lambda = 10.969 \\
&\mu = -120.309 \\
&V_{min} = 2947.937\ cm^3
\end{aligned} \tag{10}$$

and

$$\begin{aligned}
&x = 11.396\ cm \\
&y = 11.396\ cm \\
&z = 27.208\ cm \\
&\lambda = 5.698 \\
&\mu = -32.468 \\
&V_{max} = 3533.544\ cm^3
\end{aligned} \tag{11}$$