3450:335 Ordinary Differential Equations, Kreider

Interpretation of Spring Mass Systems

Given the equation of motion of a spring mass system, describe the physical characteristics of the system.

- $x(t) = 3e^{-t}\sin(4t) - e^{-t}\cos(4t)$. This is an undriven system because there is no particular solution – this is $c_1x_1(t) + c_2x_2(t)$. The system has damping because of the exponential, so the ODE has the form $x'' + \beta x' + \omega^2 x = 0$.

- $x(t) = 3\sin(2t) + 15\cos(2t)$. This is simple harmonic motion – no external force and no friction. The natural frequency is $\omega = 2$, so the ODE is $x'' + 4x = 0$.

- $x(t) = 7e^{-2t}\sin(5t) - 2e^{-2t}\cos(5t) + 2\sin(3t) - \cos(3t)$. This is a driven system with damping. The driving force must have the structure $f(t) = A\sin(3t) + B\cos(3t)$ because the particular solution is $x_p(t) = 2\sin(3t) - \cos(3t)$. The driving frequency is $M = 3$. The damping is indicated by the exponential. There is no resonance. The two indicators are (i) damped systems cannot resonate, (ii) the particular solution does not have a $t\sin(3t)$ or $t\cos(3t)$ structure. The transient solution is $7e^{-2t}\sin(5t) - 2e^{-2t}\cos(5t)$ and the steady state solution is $2\sin(3t) - \cos(3t)$.

- $x(t) = 8\sin(3t) - 2\cos(3t) + 6t\sin(3t)$. This is a resonant system. There is no damping (no exponential) and the particular solution $6t\sin(3t)$ has the same frequency as the complementary solution. The main clue is the coefficient t in the particular solution.

NOTES

1. If you remove the damping from a system, the 'natural frequency' will change. For example, $x'' + 2x' + 16x = 0$ has the solution $x(t) = c_1e^{-t}\sin(\sqrt{15}t) + c_2e^{-t}\cos(\sqrt{15}t)$, but $x'' + 16x = 0$ has solution $x(t) = c_1\sin(4t) + c_2\cos(4t)$.

2. A forced system with damping, of the form $x'' + Ax' + Bx = \sin(Mt)$, cannot resonate. The fundamental solutions have an exponential factor but the particular solution has the non-exponential structure $A\sin(Mt) + B\cos(Mt)$, so there is no chance of 'duplication' in the undetermined coefficient process.

3. A resonant system does not have a transient solution and a steady solution.

HOMEWORK SUMMARY

HW 31 has simple harmonic motion (undamped, undriven). Solve it using the technique of section 4.3.

HW 32 has a damped undriven system. Solve it via section 4.3.

HW 33 involves interpreting a solution. Here, the critical point of $x(t)$ occurs at a negative time, which is outside the domain. Hence, the maximum displacement occurs at time $t = 0$.

HW 34 has a resonant system. Solve it via undetermined coefficients. There is duplication in the standard form of the particular solution – this is the mathematical indication of resonance. You need to multiply the standard form of x_p by t; this is what causes the amplitude to increase with time – this is the physical indication of resonance.