Test Total

Name ________________________________

Exam 3 Ordinary Differential Equations Dr. Kreider

23 Nov 2009 For full credit, show your work and use correct notation

1. Solve the initial value problem $y' + 4y = 1$, $y(0) = 937$ using Laplace Transforms.

15 pts

$$(sY - 937) + 4Y = \frac{1}{s}$$

$$Y = \frac{937}{s+4} + \frac{1}{s(s+4)}$$

$$= \frac{937}{s+4} + \left[\frac{1/4}{s} - \frac{1/4}{s+4} \right]$$

$$y(t) = 937e^{-4t} + \frac{1}{4} - \frac{1}{4}e^{-4t}$$

$$= \left(937 - \frac{1}{4}\right)e^{-4t} + \frac{1}{4}$$

$$\frac{3747}{4}$$

2. Solve the initial value problem $y'' + 4y' + 29y = 0$, $y(0) = 3, y'(0) = 4$ using Laplace Transforms.

15 pts

$$(s^2 Y - 3s - 4) + 4(sY - 3) + 29Y = 0$$

$$(s^2 + 4s + 29)Y = 3s + 16$$

$$Y = \frac{3s+16}{(s+2)^2 + 25} = \frac{3(s+2) + 10}{(s+2)^2 + 25}$$

$$y(t) = 3e^{-2t}\cos 5t + 2e^{-2t}\sin 5t$$

3. Solve the initial value problem $y'' + 9y = \sin(3t)$, $y(0) = 1, y'(0) = 0$ using Laplace Transforms.

#22 $\quad t\sin(kt) <-> \dfrac{2ks}{(s^2+k^2)^2}$

#25 $\quad \sin(kt) - kt\cos(kt) <-> \dfrac{2k^3}{(s^2+k^2)^2}$

$k = 3$

15 pts

$$(s^2Y - s) + 9Y = \frac{3}{s^2+9}$$

$$Y = \frac{s}{s^2+9} + \frac{3}{(s^2+9)^2} \cdot \frac{54}{54}$$

$$y(t) = \cos 3t + \frac{1}{18}\left[\sin 3t - 3t\cos 3t\right]$$

4. Solve the initial value problem $y'' + 3y' + 2y = 1 - U(t-2)$, $y(0) = 1, y'(0) = 0$ using Laplace Transforms.

20 pts

$$(s^2 Y - s) + 3(sY - 1) + 2Y = \frac{1}{s} - \frac{1}{s}e^{-2s}$$

$$\underset{(s+2)(s+1)}{(s^2+3s+2)}\, Y = s+3 + \frac{1}{s} - \frac{1}{s}e^{-2s}$$

$$Y = \frac{s+3}{(s+2)(s+1)} + \frac{1}{s(s+2)(s+1)} - e^{-2s}\frac{1}{s(s+2)(s+1)}$$

do this once to be efficient

$$\frac{s+3}{(s+2)(s+1)} = \frac{A}{s+2} + \frac{B}{s+1} \qquad s+3 = A(s+1) + B(s+2)$$

$$s=-1 \qquad 2 = 0 + B$$

$$s=-2 \qquad 1 = -A + 0$$

$$= \frac{-1}{s+2} + \frac{2}{s+1} \longrightarrow -e^{-2t} + 2e^{-t}$$

$$\frac{1}{s(s+2)(s+1)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+1} \qquad 1 = A(s+2)(s+1) + Bs(s+1) + Cs(s+2)$$

$$s=0 \qquad 1 = 2A + 0 + 0 \qquad A = 1/2$$

$$s=-1 \qquad 1 = 0 + 0 + C(-1)(1) \qquad C = -1$$

$$s=-2 \qquad 1 = 0 + B(-2)(-1) + 0 \qquad B = 1/2$$

$$= \frac{1/2}{s} + \frac{1/2}{s+2} - \frac{1}{s+1} \longrightarrow \frac{1}{2} + \frac{1}{2}e^{-2t} - e^{-t}$$

so

$$y(t) = \left[-e^{-2t} + 2e^{-t}\right] + \left[\frac{1}{2} + \frac{1}{2}e^{-2t} - e^{-t}\right]$$

$$- \left[\frac{1}{2} + \frac{1}{2}e^{-2(t-2)} - e^{-(t-2)}\right] \mathcal{U}(t-2)$$

$$= \frac{1}{2} - \frac{1}{2}e^{-2t} + e^{-t} - [\quad]\,\mathcal{U}(t-2)$$

5. Solve the initial value problem $y'' - 6y' - 7y = 4\delta(t-3)$, $y(0) = 0, y'(0) = 8$ using Laplace Transforms.

15 pts

$$(s^2Y - 8) - 6(sY - 0) - 7Y = 4e^{-3s}$$

$$\underbrace{(s^2 - 6s - 7)}_{(s-7)(s+1)} Y = 8 + 4e^{-3s}$$

$$Y = \frac{8}{(s-7)(s+1)} + \frac{4}{(s-7)(s+1)} e^{-3s}$$

do it once

$$\frac{1}{(s-7)(s+1)} = \frac{A}{s-7} + \frac{B}{s+1}$$

$$1 = A(s+1) + B(s-7)$$

$s = 7$ $\quad 1 = 8A + 0$ $\qquad A = 1/8$

$s = -1$ $\quad 1 = 0 + B(-8)$ $\qquad B = -1/8$

$$= \frac{1/8}{s-7} + \frac{B}{s+1} \quad (*)$$

so

$$Y = 8(*) - 4(*)e^{-3s} = \frac{1}{s-7} - \frac{1}{s+1} + \frac{1}{2}\left(\frac{1}{s-7} - \frac{1}{s+1}\right)e^{-3s}$$

$$y(t) = e^{7t} - e^{-t} + \frac{1}{2}\left(e^{7(t-3)} - e^{-(t-3)}\right)u(t-3)$$

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6. Solve the initial value problem using Laplace Transforms:

$$\begin{aligned} x'(t) &= 3x(t) - 4y(t) \quad x(0) = 3 \\ y'(t) &= 4x(t) - 7y(t) \quad y(0) = 9 \end{aligned}$$

For reference, the inverse of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

20 pts

$sX - 3 = 3X - 4Y$

$sY - 9 = 4X - 7Y$

$$\begin{bmatrix} (s-3) & 4 \\ -4 & (s+7) \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \end{bmatrix}$$

$\det = (s-3)(s+7) + 16 = s^2 + 4s - 21 + 16 = s^2 + 4s - 5$

$= (s+5)(s-1)$

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \frac{1}{(s+5)(s-1)}\begin{bmatrix} s+7 & -4 \\ 4 & s-3 \end{bmatrix}\begin{bmatrix} 3 \\ 9 \end{bmatrix}$$

$$= \frac{1}{(s+5)(s-1)}\begin{bmatrix} 3(s+7) - 36 \\ 12 + 9(s-3) \end{bmatrix} = \frac{1}{(\ )(\ )}\begin{bmatrix} 3s-15 \\ 9s-15 \end{bmatrix}$$

$X = \frac{3s-15}{(s+5)(s-1)} = \frac{A}{s+5} + \frac{B}{s-1}$ $\quad 3s - 15 = A(s-1) + B(s+5)$

$s = 1 \quad -12 = 0 + 6B \quad B = -2$

$s = -5 \quad -30 = -6A + 0 \quad A = 5$

$X = \frac{5}{s+5} - \frac{2}{s-1}$

$\underline{x(t) = 5e^{-5t} - 2e^{t}}$

$Y = \frac{9s-15}{(s+5)(s-1)} = \frac{A}{s+5} + \frac{B}{s-1}$ $\quad 9s - 15 = A(s-1) + B(s+5)$

$s = 1 \quad -6 = 0 + 6B \quad B = -1$

$s = -5 \quad -60 = -6A + 0 \quad A = 10$

$Y = \frac{10}{s+5} - \frac{1}{s-1}$

$\underline{y(t) = 10e^{-5t} - e^{t}}$