Test Total

Name ______________________________

Exam 2 Ordinary Differential Equations Dr. Kreider

4 Nov 09 Show your work and use correct notation

1. Solve the initial value problem $y'' + 4y' + 13y = 0$ with $y(0) = 0$, $y'(0) = 3$.

12 pts

$m^2 + 4m + 13 = 0 \qquad m = \dfrac{-4 \pm \sqrt{16-52}}{2} = -2 \pm 3i$

$y = c_1 e^{-2x} \sin 3x + c_2 e^{-2x} \cos 3x$

$0 = y(0) = 0 + c_2 \qquad \underline{c_2 = 0}$

$y = c_1 e^{-2x} \sin 3x$

$y' = -2c_1 e^{-2x} \sin 3x + 3c_1 e^{-2x} \cos 3x$

$3 = y'(0) = 0 + 3c_1 \qquad \underline{c_1 = 1}$

$$y = e^{-2x} \sin 3x$$

2. Find the general solution to $y'' + 6y' + 9y = 2e^{-3x}$ using variation of parameters.

14 pts

$m^2 + 6m + 9 = 0 \qquad m = -3, -3$

$y_1 = e^{-3x} \qquad y_2 = xe^{-3x}$

$W = \begin{vmatrix} e^{-3x} & xe^{-3x} \\ -3e^{-3x} & e^{-3x} - 3xe^{-3x} \end{vmatrix} = e^{-6x} - 3xe^{-6x} - (-3xe^{-6x}) = e^{-6x}$

$u_1' = -\dfrac{fy_2}{W} = \dfrac{-2e^{-3x} \cdot xe^{-3x}}{e^{-6x}} = -2x \qquad u_1 = -x^2$

$u_2' = +\dfrac{fy_1}{W} = \dfrac{2e^{-3x} \cdot e^{-3x}}{e^{-6x}} = 2 \qquad u_2 = 2x$

$y = c_1 e^{-3x} + c_2 xe^{-3x} + \left[-x^2 \cdot e^{-3x} + 2x \cdot xe^{-3x}\right]$

$= c_1 e^{-3x} + c_2 xe^{-3x} + x^2 e^{-3x}$

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3. Find the general solution to $y'' + 3y' + 2y = 20\sin 2t$ using undetermined coefficients.

t not x

14 pts

$m^2 + 3m + 2 = 0$

$(m+1)(m+2) = 0 \qquad m = -1, -2$

$y_1 = e^{-t} \qquad y_2 = e^{-2t}$

standard $y_p = A\sin 2t + B\cos 2t$ no duplication

let $S = \sin 2t$, $C = \cos 2t$

$y_p' = 2AC - 2BS$

$y_p'' = -4AS - 4BC$

$[-4AS - 4BC] + 3[2AC - 2BS] + 2[AS + BC] = 20S$

$$\begin{array}{l} S(-4A - 6B + 2A) \\ + C(-4B + 6A + 2B) \end{array} = \begin{array}{l} 20S \\ + 0C \end{array}$$

$-2A - 6B = 20$

$[\;6A - 2B = 0\;] * (-3) \Rightarrow$

$-2A - 6B = 20$

$-18A + 6B = 0$

$-20A = 20$

$A = -1$

$B = -3$

$y = c_1 e^{-t} + c_2 e^{-2t} - \sin 2t - 3\cos 2t$

4. One solution to $x^2y'' - 5xy' + 9y = 0$ is $y_1 = x^3$. Use reduction of order to find the second linearly independent solution y_2. You may use the formula directly, or derive y_2 from the ODE, as you choose. You know the answer already since you recognize the equation; the point is to go through the derivation.

10 pts

$u = \int \frac{e^{-\int P dx}}{y_1^2}\, dx$

$y'' - \frac{5}{x}y' + \frac{9}{x^2}y = 0$

$\hookrightarrow P(x) = -\frac{5}{x}$

$\int P\, dx = -5\ln x = \ln x^{-5}$

$e^{-\int P dx} = e^{+5\ln x} = e^{\ln x^5} = x^5$

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$u = \int \frac{x^5}{(x^3)^2}\, dx = \int \frac{1}{x}\, dx = \ln x$

so $y_2 = u y_1$

$= x^3 \ln x$

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5. Find the general solution to $x^2y'' - 5xy' + 34y = 0$.

12 pts

$$m(m-1) - 5m + 34 = 0$$

$$m^2 - 6m + 34 = 0$$

$$m = \frac{6 \pm \sqrt{36 - 136}}{2} = 3 \pm 5i$$

$$y = c_1 x^3 \sin(5\ln x) + c_2 x^3 \cos(5\ln x)$$

6. Solve the initial value problem $\bar{\mathbf{X}}' = \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix} \bar{\mathbf{X}}$ with $\bar{\mathbf{X}}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

14 pts

$$A - \lambda I = \begin{pmatrix} 2-\lambda & -3 \\ 1 & -2-\lambda \end{pmatrix} \qquad \det = (2-\lambda)(-2-\lambda) + 3$$

$$= -4 + 2\lambda - 2\lambda + \lambda^2 + 3$$

$$= \lambda^2 - 1 = 0 \qquad \lambda = +1, -1$$

$$\lambda_1 = 1 \quad \begin{pmatrix} 1 & -3 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad k_1 - 3k_2 = 0 \qquad k_1 = 3k_2$$

$$\bar{K}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\lambda_2 = -1 \quad \begin{pmatrix} 3 & -3 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad 3k_1 - 3k_2 = 0 \qquad k_1 = k_2$$

$$\bar{K}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\bar{x}(t) = c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}$$

IC

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{3-1}\begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2-1 \\ -2+3 \end{pmatrix} = \begin{pmatrix} 1/2 \\ +1/2 \end{pmatrix}$$

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$$\bar{x}(t) = \frac{1}{2}\begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{t} + \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}$$

$$= \begin{pmatrix} \frac{3}{2}e^{t} - \frac{1}{2}e^{-t} \\ \frac{1}{2}e^{t} - \frac{1}{2}e^{-t} \end{pmatrix}$$

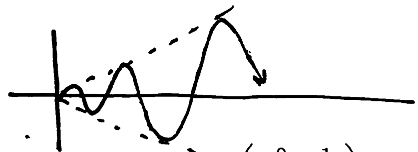
7. A spring-mass system is governed by the equation $x'' + 4x = \cos 2t$. DO NOT SOLVE THE EQUATION!
   (a) Does the system exhibit resonance? Justify your answer carefully in one or two sentences.

$M = 2$ driving freq

$w^2 = 4$ so $w = 2$ natural freq

There is resonance since $w = M$

   (b) Draw a **rough qualitative** sketch of the general solution if the initial conditions are $x(0) = 0$, $x'(0) = 1$.



10 pts

8. Find the general solution to $\bar{X}' = \begin{pmatrix} 0 & 1 \\ -9 & 6 \end{pmatrix} \bar{X}$.

14 pts

$$A - \lambda I = \begin{pmatrix} -\lambda & 1 \\ -9 & 6-\lambda \end{pmatrix} \qquad \det = (-\lambda)(6-\lambda) + 9$$

$$= \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2$$

$$\lambda = 3, 3$$

$\lambda = 3$ $\quad (A - \lambda I)\bar{K} = \bar{0}$

$$\begin{pmatrix} -3 & 1 \\ -9 & 3 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad -3k_1 + k_2 = 0 \qquad k_2 = 3k_1$$

$$\bar{K} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$(A - \lambda I)\bar{P} = \bar{K}$

$$\begin{pmatrix} -3 & 1 \\ -9 & 3 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \qquad -3p_1 + p_2 = 1$$

if $p_1 = 0$ then $p_2 = 1$

$$\bar{P} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\bar{X}(t) = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{3t} + c_2 \left[\begin{pmatrix} 1 \\ 3 \end{pmatrix} te^{3t} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{3t} \right]$$

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