3450:335 Ordinary Differential Equations, Kreider

Higher Order Constant Coefficient Linear Homogeneous Equations

The theory for higher order equations is a direct generalization of the theory for second order equations. There are 2 main features: (1) for an n^{th} order equation, there are n fundamental solutions obtained from the characteristic equation, (2) for repeated roots, the reduction of order approach works in a nice, intuitive way.

Example 1: $y''''' - y' = 0$. The characteristic equation is $m^5 - m = 0$, or $m(m^4 - 1) = 0$, which factors to $m(m^2 - 1)(m^2 + 1) = 0$. This gives $m = 0, +1, -1, +i, -i$, so the solution is (using $e^{0x} = 1$)

$$y(x) = c_1 + c_2 e^x + c_3 e^{-x} + c_4 \sin x + c_5 \cos x$$

Example 2: $y''' + 2y'' - 15y' + 14y = 0$. The characteristic equation is $m^3 + 2m^2 - 15m + 14 = 0$. This is hard to factor but it turns out to be $(m - 2)(m^2 + 4m - 7) = 0$. Using the quadratic formula, the 3 roots are $m = 2, -2 \pm \sqrt{11}$ so the solution is

$$y(x) = c_1 e^{2x} + c_2 e^{(-2+\sqrt{11})x} + c_3 e^{(-2-\sqrt{11})x}$$

Example 3: $y'''' - 12y''' + 54y'' - 108y' + 81y = 0$. The characteristic equation is $m^4 - 12m^3 + 54m^2 - 108y' + 81 = 0$, which reduces to $(m - 3)^4 = 0$. The single root $m = 3$ is repeated 4 times. The 4 fundamental solutions follow a beautiful pattern derived from reduction of order:

$$y(x) = c_1 e^{3x} + c_2 x e^{3x} + c_3 x^2 e^{3x} + c_4 x^3 e^{3x}$$

Example 4: $y'''' - 4y''' + 8y'' - 8y' + 4y = 0$. The characteristic equation is $m^4 - 4m^3 + 8m^2 - 8m + 4 = 0$, which reduces to $(m^2 - 2m + 1)^2 = 0$. Set $m^2 - 2m + 1 = 0$ and use the quadratic formula to get $m = -1 \pm i$. These roots are repeated, and the reduction of order formula applies as you would think:

$$y(x) = k_1 e^{-x} \sin x + k_2 e^{-x} \cos x + k_3 x e^{-x} \sin x + k_4 x e^{-x} \cos x$$