3450:335 Ordinary Differential Equations, Kreider

Exact Equations

Here is an approach to exact differential equations that is slightly different from that in the textbook. To begin, let's go backwards – start with the solution and see what the corresponding ODE is.

Consider the level curve $f(x, y) = x^2y^3 + 2x + 5y^2 = c$. Take the total differential of both sides to get $df = 0$, which is $f_x dx + f_y dy = 0$, or

$$(2xy^3 + 2)dx + (3x^2y^2 + 10y)dy = 0 \tag{1}$$

So, if we started with the ODE (1), we can recover the function $f(x, y)$ by integrating the 2 partial derivatives. Here is the process: We know that $f_x = 2xy^3 + 2$, so we integrate with respect to x to get

$$f(x, y) = x^2y^3 + 2x + g(y). \tag{2}$$

We also have that $f_y = 3x^2y^2 + 10y$, so we integrate with respect to y to get

$$f(x, y) = x^2y^3 + 5y^2 + h(x). \tag{3}$$

Now we have 2 representations of the same function, so they must be equal:

$$x^2y^3 + 2x + g(y) = f(x, y) = x^2y^3 + 5y^2 + h(x) \tag{4}$$

From this, we can see that $g(y) = 5y^2$ and $h(x) = 2x$. The complete function is

$$f(x, y) = x^2y^3 + 2x + 5y^2 \tag{5}$$

and the solution to the ODE is $x^2y^3 + 2x + 5y^2 = c$.

The interpretation is that $g(y)$ is the part of $f(x, y)$ that was lost when differentiating with respect to x and $h(x)$ is the part of $f(x, y)$ that was lost when differentiating with respect to y. By looking at both integrals, we can see all of f.