3450:335 Ordinary Differential Equations, Kreider

Typical types of Exam 3 problems

1. Standard Partial Fractions

   $y'' + 10y' + 9y = 1$ with $y(0) = 2$, $y'(0) = 3$.

$$[s^2Y - 2s - 3] + 10[sY - 2] + 9[Y] = \frac{1}{s}$$

$$(s^2 + 10s + 9)Y - 2s - 3 - 20 = \frac{1}{s}$$

$$(s+1)(s+9)Y = 2s + 23 + \frac{1}{s}$$

$$Y = \frac{2s+23}{(s+1)(s+9)} + \frac{1}{s(s+1)(s+9)}$$

$$\frac{2s+23}{(s+1)(s+9)} = \frac{A}{s+1} + \frac{B}{s+9}$$

$$2s + 23 = A(s+9) + B(s+1)$$

$s = -9$ $\quad -18 + 23 = 0 - 8B$

$\quad +5 = -8B \qquad B = -5/8$

$s = -1$ $\quad -2 + 23 = A \cdot 8 + 0$

$\quad +21 = 8A \qquad A = +21/8$

$$\frac{1}{s(s+1)(s+9)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+9}$$

$$1 = A(s+1)(s+9) + Bs(s+9) + Cs(s+1)$$

$s = 0$ $\quad 1 = A(1)(9) + 0 + 0$

$\quad A = 1/9$

$s = -1$ $\quad 1 = 0 + B(-1)(8) + 0$

$\quad B = -1/8$

$s = -9$ $\quad 1 = 0 + 0 + C(-9)(-8)$

$\quad C = 1/72$

$$Y(s) = \frac{+21/8}{s+1} - \frac{5/8}{s+9} + \frac{1/9}{s} - \frac{1/8}{s+1} + \frac{1/72}{s+9}$$

$$y(t) = +\frac{21}{8}e^{-t} - \frac{5}{8}e^{-9t} + \frac{1}{9} - \frac{1}{8}e^{-t} + \frac{1}{72}e^{-9t}$$

$$= \frac{5}{2}e^{-t} - \frac{11}{18}e^{-9t} + \frac{1}{9}$$

2. Complete the square

$y'' + 6y' + 57y = 0$ with $y(0) = 1$, $y'(0) = 2$.

$$[s^2Y - s - 2] + 6[sY - 1] + 57[Y] = 0$$

$$(s^2 + 6s + 57)Y = s + 8$$

$$s^2 + 6s + 9 + 48$$

$$(s+3)^2 + 48$$

$$Y = \frac{s+8}{(s+3)^2+48} = \frac{s+3}{(s+3)^2+48} + \frac{5}{(s+3)^2+48}\left(\cdot \frac{\sqrt{48}}{\sqrt{48}}\right)$$

$$y(t) = e^{-3t}\cos(\sqrt{48}\,t) + \frac{5}{\sqrt{48}} \sin(\sqrt{48}\,t)$$

e^{-3t}

3. Duplication in y_p (formulas 22, 25)

$y'' + 4y = \cos 2t$ with $y(0) = 7$, $y'(0) = 0$.

$$[s^2Y - 7s - 0] + 4[Y] = \frac{s}{s^2+4}$$

$$(s^2+4)Y = 7s + \frac{s}{s^2+4}$$

$$Y = \frac{7s}{s^2+4} + \frac{s}{(s^2+4)^2} \cdot \frac{4}{4}$$

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need $2k = 2 \cdot 2 = 4$

$$y(t) = 7\cos 2t + \frac{1}{4} t \sin 2t$$

4. Integral formula

$y'' + y = 1$ with $y(0) = 2$, $y'(0) = 1$. This one uses the integral formula at the end (alternative to partial fractions) – it's optional

$$[s^2Y - 2s - 1] + [Y] = \frac{1}{s}$$

$$(s^2+1)Y = 2s+1 + \frac{1}{s}$$

$$Y = \frac{2s+1}{s^2+1} + \frac{1}{s(s^2+1)}$$

$$\frac{1}{s} \cdot \frac{1}{s^2+1}$$

integral, $\sin t$ so $\rightarrow \int_0^t \sin\tau \, d\tau$

$$= -\cos\tau \Big|_0^t$$

$$= -\cos t + 1$$

$$2\frac{s}{s^2+1} + \frac{1}{s^2+1}$$

$$y(t) = 2\cos t + \sin t - \cos t + 1$$

$$= \cos t + \sin t + 1$$

$y' + 6\int_0^t y(\tau)\,d\tau = 2$ with $y(0) = 1$. Here, you need the integral formula at the beginning.

$$[sY - 1] + 6\left[\frac{1}{s}Y\right] = \frac{2}{s}$$

$$\left(s + \frac{6}{s}\right)Y = 1 + \frac{2}{s} \qquad \text{multiply by } s$$

$$(s^2+6)Y = s+2$$

$$Y = \frac{s+2}{s^2+6} = \frac{s}{s^2+6} + \frac{2}{s^2+6} \cdot \frac{\sqrt{6}}{\sqrt{6}}$$

$$y(t) = \cos\sqrt{6}\,t + \frac{2}{\sqrt{6}}\sin\sqrt{6}\,t$$

5. Heaviside function (unit step function)

$y' + y = tU(t-3)$ with $y(0) = 2345$

Source $\quad t\,\mathcal{U}(t-3) = (t-3+3)\,\mathcal{U}(t-3)$
$= (t-3)\,\mathcal{U}(t-3) + 3\,\mathcal{U}(t-3)$

unshifted: $\quad f(t) = t \rightarrow \frac{1}{s^2} \qquad f(t) = 3 \rightarrow \frac{3}{s}$

ODE $\quad sY(s) - 2345 + Y(s) = \left(\frac{1}{s^2} + \frac{3}{s}\right) e^{-3s}$

$(s+1)\,Y(s) = 2345 + \quad ''$

$Y(s) = \frac{2345}{s+1} + \left(\frac{1}{s^2(s+1)} + \frac{3}{s(s+1)}\right) e^{-3s}$

combine in a single PF

$\frac{1}{s^2(s+1)} + \frac{3s}{s^2(s+1)} = \frac{3s+1}{s^2(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1}$

$3s+1 = As(s+1) + B(s+1) + Cs^2$

$s=0$	$0+1 = 0 + B + 0$	$B = 1$
$s=-1$	$-3+1 = 0 + 0 + C$	$C = -2$
$s=1$	$3+1 = 2A + 2B + C$	
	$4 = 2A + 2 - 2$	$A = 2$

$Y(s) = \frac{2345}{s+1} + \left(\frac{2}{s} + \frac{1}{s^2} - \frac{2}{s+1}\right) e^{-3s}$

unshifted: $\quad 2 + t - 2e^{-t}$

$y(t) = 2345e^{-t} + \left(2 + [t-3] - 2e^{-[t-3]}\right)\mathcal{U}(t-3)$

6. Delta function

$$y'' + 4y' + 4y = \delta(t-72) \text{ with } y(0) = 1,\ y'(0) = 0$$

$$[s^2Y - s - 0] + 4[sY - 1] + 4[Y] = e^{-72s}$$

$$(s^2+4s+4)Y = s+4 + e^{-72s}$$

$$(s+2)^2$$

$$Y = \frac{s+4}{(s+2)^2} + \frac{1}{(s+2)^2} e^{-72s}$$

$$= \frac{s+2+2}{(s+2)^2} + \frac{1}{(s+2)^2} e^{-72s}$$

$$= \frac{1}{s+2} + \frac{2}{(s+2)^2} + \frac{1}{(s+2)^2} e^{-72s}$$

$$\frac{2}{s^2} \to 2t \qquad \frac{1}{s^2} \to t$$

$$\frac{2}{(s+2)^2} \to 2t\,e^{-2t} \qquad \frac{1}{(s+2)^2} \to te^{-2t}$$

unshifted

$$y(t) = e^{-2t} + 2t\,e^{-2t} + (t-72)\,e^{-2(t-72)}\,\mathcal{U}(t-72)$$

7. Systems

$$\begin{aligned} x' &= 5x - 3y \\ y' &= 2x + y \end{aligned}$$

with $x(0) = -2$, $y(0) = 3$

$$\begin{aligned} sX + 2 &= 5X - 3Y \\ sY - 3 &= 2X + Y \end{aligned} \qquad \begin{bmatrix} s-5 & 3 \\ -2 & s-1 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

$$\begin{aligned} \det = (s-5)(s-1) - (-6) &= s^2 - 6s + 11 \\ &= (s-3)^2 + 2 \end{aligned}$$

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \frac{1}{(s-3)^2+2} \begin{bmatrix} s-1 & -3 \\ 2 & s-5 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \frac{1}{(s-3)^2+2} \begin{bmatrix} -2(s-1) - 9 \\ -4 + 3(s-5) \end{bmatrix}$$

$$X(s) = \frac{-2s-7}{(s-3)^2+2} = \frac{-2(s-3+3)-7}{(s-3)^2+2} = \frac{-2(s-3) - 13}{(s-3)^2+2}$$

- $x(t) = -2e^{3t}\cos\sqrt{2}t - \frac{13}{\sqrt{2}} e^{3t} \sin\sqrt{2}t$

$$Y(s) = \frac{3s-19}{(s-3)^2+2} = \frac{3(s-3+3)-19}{(s-3)^2+2} = \frac{3(s-3) - 10}{(s-3)^2+2}$$

- $y(t) = 3e^{3t}\cos\sqrt{2}t - \frac{10}{\sqrt{2}} e^{3t} \sin\sqrt{2}t$