3450:335 Ordinary Differential Equations, Kreider

Spring Mass Problems

Example 1. A mass weighing 6 pounds stretches a spring 1 foot. The mass is replaced by another weighing 16 pounds. At time $t = 0$, it is released from a position 4 inches above equilibrium with a downward velocity of 2 ft/s.

Find the equation of motion in phase form, the maximum displacement and the time at which the mass first goes through the equilibrium position.

- 1 $\quad mg = ks \qquad 6 = k \cdot 1 \qquad k = 6$
- ODE $\quad mx'' + \beta x' + kx = 0$

  $\beta = 0$ no damping $\qquad$ so $\quad \frac{1}{2}x'' + 6x = 0 \qquad \boxed{x'' + 12x = 0}$

  - $mg = 16$ so $m = \frac{16}{g} = \frac{1}{2}$
- soln $\quad x(t) = c_1 \cos(\sqrt{12}\,t) + c_2 \sin(\sqrt{12}\,t)$
- 2 IC $\qquad x(0) = -\frac{1}{3} \qquad x'(0) = +2$

$$-\frac{1}{3} = x(0) = c_1 + 0 \qquad c_1 = -\frac{1}{3}$$

$$x' = -\sqrt{12}\left(-\tfrac{1}{3}\right)\sin\sqrt{12}\,t + \sqrt{12}\,c_2\cos\sqrt{12}\,t$$

$$2 = x'(0) = 0 + \sqrt{12}\,c_2 \qquad c_2 = \frac{2}{\sqrt{12}}$$

- phase form $\quad x(t) = A\sin(\sqrt{12}\,t + \phi)$

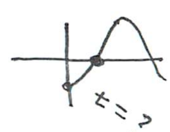
$$A^2 = c_1^2 + c_2^2 = \frac{1}{9} + \frac{4}{12} = \frac{4}{9} \qquad \text{So} \quad A = \frac{2}{3}$$

$$\cos\phi = \frac{c_2}{A} = \frac{2/\sqrt{12}}{2/3} = \frac{3}{\sqrt{12}} = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}$$

$$\sin\phi = \frac{c_1}{A} = \frac{-1/3}{2/3} = -1/2$$

$$\text{so} \quad \phi = \sin^{-1}\left(-\tfrac{1}{2}\right) = -\frac{\pi}{6}$$

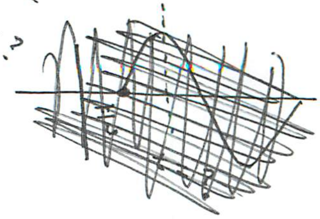
- $x(t) = \frac{2}{3}\sin\left(\sqrt{12}\,t - \frac{\pi}{6}\right)$



- max displacement : $\quad x = \frac{2}{3}$
- time to $x = 0$ is when

$$x = \frac{2}{3}\sin 0 \qquad \text{so} \qquad \sqrt{12}\,t - \frac{\pi}{6} = 0$$

$$t = \frac{\pi/6}{\sqrt{12}} = .15 \text{ sec}$$



$m = 1$ $\qquad$ $k = 15$

Example 2. A 1 kg mass is attached to a spring whose constant is 15 N/m. The system is submerged in a liquid that imparts a damping force numerically equal to 8 times the instantaneous velocity. The mass is released from rest at a point 1 m below equilibrium. Find the equation of motion. $\qquad \beta = 8$

$x'(0) = 0 \qquad x(0) = 1$

$$x'' + 8x' + 15x = 0$$

$$(m+3)(m+5)$$

$$x(t) = \underbrace{c_1}_{\frac{5}{2}} e^{-3t} + \underbrace{c_2}_{-\frac{3}{2}} e^{-5t}$$

overdamped

$m = 2$ $\qquad\qquad$ $k = 40$

Example 3. A 2 kg mass is attached to a spring whose constant is 40 N/m. The system is submerged in a liquid that imparts a damping force numerically equal to 12 times the instantaneous velocity. The mass is released from a point 1 m below equilibrium with an initial upward velocity of 1 m/s. Find the equation of motion. $x(0) = 1$

$\beta = 12$

$x'(0) = -1$

$$2x'' + 12x' + 40x = 0$$

$$x'' + 6x' + 20x = 0$$

underdamped

$$m = -3 \pm i\sqrt{11}$$

$$x(t) = \underset{\displaystyle 1}{\underset{\downarrow}{c_1}}\, e^{-3t} \cos\sqrt{11}\,t + \underset{\displaystyle \frac{2}{\sqrt{11}}}{\underset{\downarrow}{c_2}}\, e^{-3t} \sin\sqrt{11}\,t$$

$m = 3$ $\qquad\qquad$ $k = 48$

Example 4. A 3 kg mass is attached to a spring whose constant is 48 N/m. The system is submerged in a liquid that imparts a damping force numerically equal to 24 times the instantaneous velocity. Find the equation of motion.

$\beta = 24$

$$3x'' + 24x' + 48x = 0$$

$$x'' + 8x' + 16x = 0$$

$$(m+4)^2$$

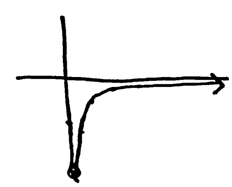
critically damped

$$x(t) = c_1 e^{-4t} + c_2 t e^{-4t}$$

i) $x(0) = 1 \quad x'(0) = \underline{\underline{2}} \longrightarrow c_1 = 1, \; c_2 = 6$

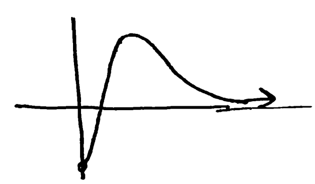
adding just a bit of initial energy leads to the slightest bit of oscillation, but the mass doesn't cross the equilibrium position

ii) $x(0) = \underline{\underline{-1}}, \quad x'(0) = 2 \longrightarrow c_1 = -1, \; c_2 = -2$



changing the initial configuration still doesn't lead to oscillation

iii) $x(0) = -1, \quad x'(0) = \underline{\underline{5}} \longrightarrow c_1 = -1, \; c_2 = 1$



adding more initial energy makes the spring a bit more oscillatory

Example 5. The system in Example 4 now has a piston attached to it, providing an external source of the form $f(t) = 6\sin 4t$, so the ODE is $x'' + 8x' + 16x = 2\sin 4t$. Find the equation of motion and identify the transient component and the steady state component.

$$x'' + 8x' + 16x = 2\sin 4t$$

$$y_p = A\sin 4t + B\cos 4t$$

$$x(t) = \underbrace{c_1 e^{-4t} + c_2 t e^{-4t}}_{\text{transient}} \underbrace{- \frac{1}{16}\cos 4t}_{\text{steady state}} \qquad \left(A = 0,\ B = -\frac{1}{16}\right)$$

Example 6. The driven system of Example 5 is removed from the damping liquid, so the ODE is $x'' + 16x = 2\sin 4t$. Find the equation of motion and identify whether or not the system exhibits resonance.

$x_p = \sin 4t \qquad x_2 = \cos 4t$

standard $x_p = A\sin 4t + B\cos 4t$

duplication, so go to $x_p = At\sin 4t + Bt\cos 4t$

soln is $x = c_1 \sin 4t + c_2 \cos 4t - \frac{1}{4} t \cos 4t$

There are several ways to identify resonance:

i) From eqn, $w^2 = 16$ so $w = 4$, and $M = 4$

ii) From technique, duplication implies resonance when right hand side is sin/cos

iii) From $x_p = -\frac{1}{4} t \cos 4t$

increases the amplitude