Test Total

Name ______________________________

Exam 2 Ordinary Differential Equations Dr. Kreider
4 Nov 13 Show your work and use correct notation

1. Solve the initial value problem $y'' + 8y' + 20y = 0$ with $y(0) = 0$, $y'(0) = -2$.

12 pts

$m^2 + 8m + 20 = 0 \qquad m = \dfrac{-8 \pm \sqrt{64 - 80}}{2}$

$= -4 \pm 2i$

$y = c_1 e^{-4x} \cos 2x + c_2 e^{-4x} \sin 2x$

$0 = y(0) = c_1 \cdot 1 + c_2 \cdot 0 \implies c_1 = 0$

$y' = -4 c_2 e^{-4x} \sin 2x + 2 c_2 e^{-4x} \cos 2x$

$-2 = y'(0) = -4c_2 \cdot 0 + 2c_2 \cdot 1 \implies c_2 = -1$

$y = -e^{-4x} \sin 2x$

2. Find the general solution to $y'' + 4y' + 4y = 6e^{-2x}$ using variation of parameters.

14 pts

$m^2 + 4m + 4 = 0 \qquad m = -2, -2$

$y_1 = e^{-2x} \qquad y_2 = xe^{-2x}$

$$W = \begin{vmatrix} e^{-2x} & xe^{-2x} \\ -2e^{-2x} & e^{-2x} - 2xe^{-2x} \end{vmatrix} = e^{-4x}$$

$u_1' = -\dfrac{f y_2}{W} = \dfrac{-6e^{-2x} \cdot xe^{-2x}}{e^{-4x}} = -6x \qquad u_1 = -3x^2$

$u_2' = +\dfrac{f y_1}{W} = \dfrac{6e^{-2x} e^{-2x}}{e^{-4x}} = 6 \qquad u_2 = 6x$

$y_p = u_1 y_1 + u_2 y_2 = -3x^2 e^{-2x} + 6x \cdot xe^{-2x} = 3x^2 e^{-2x}$

$y = c_1 e^{-2x} + c_2 xe^{-2x} + 3x^2 e^{-2x}$

Page 1: 26

3. Find the general solution to $y'' + 2y' + y = 5e^{3x}$ using undetermined coefficients.

11 pts

$m^2+2m+1=0 \quad m=-1,-1 \quad y_1 = e^{-x} \quad y_2 = xe^{-x}$

$y_p = Ae^{3x}$ — no duplication

$y_p' = 3Ae^{3x}$

$y_p'' = 9Ae^{3x}$

$[9Ae^{3x}] + 2[3Ae^{3x}] + [Ae^{3x}] = 5e^{3x}$

$16A = 5$

$y = c_1 e^{-x} + c_2 x e^{-x} + \frac{5}{16} e^{3x}$

4. Set up but do not solve. Write the differential equation and initial conditions that describe the following spring-mass system. A 2 kg object stretches a spring 1 m. The object is removed and is replaced with a 5 kg object. The spring-mass system is immersed in a liquid that imparts a resistance equal to 3 times the instantaneous velocity. The object is released from rest at a point 2 m above the equilibrium position.

10 pts

$mg = ks$

$2 \cdot 9.8 = k \cdot 1 \qquad k = 19.6$

$m = 5 \quad \beta = 3 \quad \Longrightarrow \quad 5x'' + 3x' + 19.6x = 0$

$x(0) = -2$

$x'(0) = 0$

Page 2: 24

Name ______________________________

5. Find the general solution to $x^2y'' - 3xy' + 25y = 0$.

12 pts

$m(m-1) - 3m + 25 = 0$

$m^2 - 4m + 25 = 0 \qquad m = \dfrac{4 \pm \sqrt{4^2 - 100}}{2}$

$= 2 \pm i \dfrac{\sqrt{84}}{2} = 2 \pm i\sqrt{21}$

$y = c_1 x^2 \cos(\sqrt{21} \ln x) + c_2 x^2 \sin(\sqrt{21} \ln x)$

6. Solve the initial value problem $\bar{X}' = \begin{pmatrix} 4 & 1 \\ -3 & 0 \end{pmatrix} \bar{X}$ with $\bar{X}(0) = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$.

14 pts

$A - \lambda I = \begin{pmatrix} 4-\lambda & 1 \\ -3 & -\lambda \end{pmatrix}$

$\det = (4-\lambda)(-\lambda) - (-3) = -4\lambda + \lambda^2 + 3 = (\lambda-1)(\lambda-3)$

$\lambda = 1 : \begin{pmatrix} 3 & 1 \\ -3 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad 3a + b = 0 \qquad \bar{K} = \begin{pmatrix} a \\ -3a \end{pmatrix}$

$b = -3a$

use $\bar{K} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

$\lambda = 3 \qquad \begin{pmatrix} 1 & 1 \\ -3 & -3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad a + b = 0$

$a = -b$ use $\bar{K} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\bar{X} = c_1 \begin{pmatrix} 1 \\ -3 \end{pmatrix} e^{+t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{+3t}$

$\begin{pmatrix} 4 \\ 2 \end{pmatrix} = \bar{X}(0) = c_1 \begin{pmatrix} 1 \\ -3 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \dfrac{1}{1-3} \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 2 \end{pmatrix} = \dfrac{1}{-2}\begin{pmatrix} 4+2 \\ 12+2 \end{pmatrix} = \begin{pmatrix} -3 \\ -7 \end{pmatrix}$

Page 3: 26

$\bar{X} = \begin{pmatrix} -3 \\ 9 \end{pmatrix} e^{t} + \begin{pmatrix} 7 \\ -7 \end{pmatrix} e^{3t}$

Name ______________________________

7. A spring-mass system has solution $x(t) = 3e^{-t}\sin(2t) - 2e^{-t}\cos(2t) + 5\sin(2t)$. (i) Is there damping in the system? Why or why not? (ii) Is there resonance in the system? Why or why not? (iii) Is there a transient component to the solution? If so, what is it? (iv) Is there a steady state component to the solution? If so, what is it?

10 pts

i) damping – yes, e^{-t}

ii) resonance – no, no dupl ($\pm$)
no resonance with damping

iii) transient : $3e^{-t}\sin 2t - 2e^{-t}\cos 2t$

iv) steady state : $5\sin 2t$

8. Find the general solution to $\bar{X}' = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}\bar{X}$.

14 pts

$A - \lambda I = \begin{pmatrix} 3-\lambda & 1 \\ -1 & 1-\lambda \end{pmatrix}$

$\det = (3-\lambda)(1-\lambda) - (-1) = 3 - \lambda - 3\lambda + \lambda^2 + 1$
$= \lambda^2 - 4\lambda + 4 = (\lambda-2)^2$
$\lambda = 2, 2$

$(A-\lambda I)\bar{K} = \bar{0}$: $\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $a + b = 0$
$\hookrightarrow \bar{K} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$(A-\lambda I)\bar{P} = \bar{K}$ $\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $a + b = 1$
$\hookrightarrow \bar{P} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\bar{X} = c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{2t} + c_2\left[\begin{pmatrix} 1 \\ -1 \end{pmatrix}te^{2t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}e^{2t}\right]$$

Page 4: 24