Test Total

Name ______________________________

Exam 1 — Ordinary Differential Equations — Dr. Kreider

26 September 2014 — For full credit, show your work and use correct notation

1. Solve this differential equation: $y\frac{dy}{dx} = xy^2 + x$ with initial condition $y(0) = 1$. Solve this explicitly for $y(x)$ instead of leaving it in implicit form.

12 pts

$= x(y^2+1)$

$$\frac{y}{y^2+1}\,dy = x\,dx$$

$u = y^2+1$

$$\frac{1}{2}\ln(y^2+1) = \frac{x^2}{2} + c$$

$$\ln(y^2+1) = x^2 + d \quad (d = 2c)$$

IC $\ln(1+1) = 0 + d$

$$d = \ln 2$$

$$\ln(y^2+1) = x^2 + \ln 2$$

$$y^2 + 1 = e^{x^2+\ln 2} = e^{\ln 2}e^{x^2} = 2e^{x^2}$$

$$y = \sqrt{2e^{x^2} - 1}$$

+ from IC : $y > 0$

2. Solve this initial value problem: $(xy + 2x^2)dx + x^2dy = 0$ with $y(1) = 3$. Solve this explictly for $y(x)$ instead of leaving it in implicit form.

13 pts

$y = ux$ — net powers $= 2$ — homogeneous

$$(xux + 2x^2)\,dx + x^2(u\,dx + x\,du) = 0$$

$$(ux^2 + 2x^2 + ux^2)\,dx + x^3\,du = 0$$

$$2x^2(u+1)\,dx + x^3\,du = 0$$

$$\frac{2}{x}\,dx = -\frac{1}{u+1}\,du$$

$$2\ln x + c = -\ln(u+1)$$ [IC implies $x>0$, $y>0$]

$$-2\ln x + c = \ln(u+1)$$

$$\ln\frac{1}{x^2} + c = \text{''}$$

$$e^{\ln\frac{1}{x^2}+c} = u+1$$

$$\frac{k}{x^2} = u + 1 = \frac{y}{x} + 1$$

$$\frac{k}{x^2} - 1 = \frac{y}{x}$$

$$\frac{k}{x} - x = y$$

$$\frac{k}{1} - 1 = 3 \qquad k = 4 \qquad y = \frac{4}{x} - x$$

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3. Solve this initial value problem: $x^2 y' - 4xy = x^6 e^x$ with $y(2) = 0$.

13 pts

linear

$y' - \frac{4}{x} y = x^4 e^x$

$P = -\frac{4}{x} \quad \int P\,dx = -4\ln x \quad k = e^{-4\ln x} = e^{\ln x^{-4}} = \frac{1}{x^4}$

$\frac{d}{dx}\left(\frac{1}{x^4} y\right) = e^x$

$\frac{1}{x^4} y = e^x + c$

IC $\quad 0 = e^2 + c$

$c = -e^2$

$y = x^4 e^x + x^4 c = x^4(e^x - e^2)$

4. At the Zippy Pizza Plaza, pizzas are prepared in a cool room with ingredients at 60^o (T_0) and placed in a 1000^o (T_m) oven to bake. After 5 minutes, the pizza's temperature is 95^o. How many minutes will it take the pizza to reach the serving temperature of 160^o? Newton's Law of Cooling is $T(t) = T_m + (T_0 - T_m)e^{-kt}$.

12 pts

$T = 1000 + (60 - 1000)e^{-kt} = 1000 - 940 e^{-kt}$

$95 = T(5) = 1000 - 940 e^{-5k}$

$-905 = -940 e^{-5k}$

$\frac{905}{940} = e^{-5k} \qquad -5k = \ln\left(\frac{905}{940}\right) \qquad k = \frac{1}{5}\ln\left(\frac{940}{905}\right)$

$= .0076$

$160 = 1000 - 940 e^{-kt}$

$\frac{840}{940} = e^{-kt}$

$-kt = \ln\frac{840}{940}$

$t = \frac{1}{k}\ln\frac{940}{840} = 14.82$ min

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5. Solve the exact problem $(ye^{xy} - y\sin x + 3x^2)dx + (xe^{xy} + \cos x - 16y^{15})dy = 0$. You do not need to verify that it is exact.

12 pts

$$f = \int M\,dx + g(y) = \int ye^{xy} - y\sin x + 3x^2\,dx + g(y)$$
$$= e^{xy} + y\cos x + \underline{x^3} + \underline{\underline{g(y)}} \quad (*)$$

$$N = \frac{\partial f}{\partial y}$$
$$xe^{xy} + \cos x - 16y^{15} = xe^{xy} + \cos x + 0 + g'(y)$$
$$-16y^{15} = g' \quad \text{so} \quad g = -y^{16}$$

or

$$f = \int N\,dy + h(x) = \int xe^{xy} + \cos x - 16y^{15}\,dy + h(x)$$
$$= e^{xy} + y\cos x \underline{\underline{-y^{16}}} + \underline{h(x)}$$

(compare with $*$)

soln $f(x,y) = c$

$$e^{xy} + y\cos x + x^3 - y^{16} = c$$

6. On the planet Mongo, the gravitational constant is $g = 100$ m/s^2 and the atmosphere imparts an air resistance that is proportional to velocity with proportionality constant 5 kg/s. Suppose a block with mass 1 kg is thrown from a 20 m tall platform with initial velocity $v_0 = 3$ m/s. Set up and solve the ODE for the velocity of the block.

13 pts

$$m\frac{dv}{dt} = mg - kv \qquad m = 1, \; g = 100, \; k = 5$$
$$\frac{dv}{dt} = 100 - 5v$$

i) $\frac{dv}{dt} + 5v = 100$

$\underset{IF}{\mu} = e^{5t}$

$\frac{d}{dt}(e^{5t}v) = 100e^{5t}$

$e^{5t}v = 20e^{5t} + c$

$v(0) = 3$ $3 = 20 + c$

$-17 = c$

$v = 20 - 17e^{-5t}$

ii) $\frac{dv}{100 - 5v} = dt$

$-\frac{1}{5}\ln(100 - 5v) = t + c$

$\ln(100 - 5v) = -5t + d \quad (d = -5c)$

$100 - 5v = \alpha e^{-5t}$

IC $100 - 15 = \alpha = 85$

$100 - 85e^{-5t} = 5v$

$20 - 17e^{-5t} = v$

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Name $V = 100$

7. A 100 cubic meter tank is filled with a salt solution with concentration 73 kg/m^3. An inflow pipe carries a salt solution with concentration 1 kg/m^3 into the tank at the rate of 6 m^3/min. An outflow pipe carries the well-mixed solution out of the tank at the same rate. Set up and solve the initial value problem to find the salt concentration $S(t)$ in the tank.

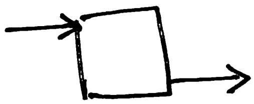
$S(0) = 7300$ kg

a little tricky, so it was ok if you used 73

12 pts

$r = 6$

$c_{in} = 1$



$r = 6$ $c = \frac{S}{V}$

$$\frac{dS}{dt} = rc_{in} - \frac{r}{V}S = 6 - .06S$$

$$\frac{dS}{dt} + .06S = 6$$

$\hookrightarrow k = e^{.06t}$

$$\frac{d}{dt}(e^{.06t}S) = 6e^{.06t}$$

$\frac{6}{.06}e^{.06t}$

$$e^{.06t}S = 100e^{.06t} + c$$

$$S = 100 + ce^{-.06t}$$

$7300 = S(0) = 100 + c$ $c = +7200$ $S = 100 + 7200e^{-.06t}$

or

$73 = S(0) = 100 + c$ $c = -27$ $S = 100 - 27e^{-.06t}$

8. Solve the initial value problem $y' - y = e^{2x}y^{-3}$ with $y(0) = 1$.

Bernoulli: $n = -3$ $u = y^{1-n} = y^4$

$u' = 4y^3y'$

$\frac{1}{4y^3}u' = y'$

13 pts

$$\frac{1}{4y^3}u' - y = e^{2x}y^{-3}$$

$$u' - 4y^4 = 4e^{2x}$$

$\hookrightarrow p = -4$ $k = e^{-4x}$

$$\frac{d}{dx}(e^{-4x}u) = 4e^{-2x}$$

$\{4e^{2x} \cdot e^{-4x}\}$

$$e^{-4x}u = -2e^{-2x} + c$$

$$u = -2e^{+2x} + ce^{4x}$$

IC $1^4 = -2 + c$ $c = 3$

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$$u = -2e^{2x} + 3e^{4x}$$

$$y = (-2e^{2x} + 3e^{4x})^{1/4}$$