3450:335 Ordinary Differential Equations, Kreider

Partial fraction forms

Partial fractions are used to decompose a complicated fraction into simpler pieces. The degree of the numerator must be strictly less than the degree of the denominator. Since the focus here is on the denominators, the numerator is set to 1 for simplicity.

- Linear Factors. The basic structure is "constant over linear".

$$\begin{aligned}\frac{1}{(s-1)(s-2)} &= \frac{A}{s-1}+\frac{B}{s-2}\\ \frac{1}{s(s+6)} &= \frac{A}{s}+\frac{B}{s+6}\end{aligned}$$

- Quadratic Factors. When a quadratic expression doesn't factor, the basic structure is "linear over quadratic".

$$\begin{aligned}\frac{1}{(s-1)(s^2+2)} &= \frac{A}{s-1}+\frac{Bs+C}{s^2+2}\\ \frac{1}{(s^2+3)(s^2+5)} &= \frac{As+B}{s^2+3}+\frac{Cs+D}{s^2+5}\\ \frac{1}{(s-1)(s^2+2s+2)} &= \frac{A}{s-1}+\frac{Bs+C}{s^2+2s+2}=\frac{A}{s-1}+\frac{Bs+C}{(s+1)^2+1}\end{aligned}$$

- Repeated Linear Factors. When a linear factor is repeated, there is a term for each repetition. Think of an s as $s-0$, and s^2 by itself as $(s-0)^2$.

$$\begin{aligned}\frac{1}{(s-1)^2(s-2)} &= \frac{A}{s-1}+\frac{B}{(s-1)^2}+\frac{C}{s-2}\\ \frac{1}{(s-1)^3(s-2)} &= \frac{A}{s-1}+\frac{B}{(s-1)^2}+\frac{C}{(s-1)^3}+\frac{D}{s-2}\\ \frac{1}{s^2(s+8)} &= \frac{A}{s}+\frac{B}{s^2}+\frac{C}{s+8}\\ \frac{1}{s(s-1)^2(s^2+2)} &= \frac{A}{s}+\frac{B}{s-1}+\frac{C}{(s-1)^2}+\frac{Ds+E}{s^2+2}\end{aligned}$$

  Note that in the first example, the first 2 terms could be combined into $\dfrac{A(s-1)+B}{(s-1)^2}$, which fits the "linear over quadratic" template. However, splitting it into 2 terms is easier for using Laplace Transforms.