FIRST ORDER - 5 techniques 4 questions

$M\,dx + N\,dy = 0$ exact or homogeneous

$a(x)y' + b(x)y = c(x)$ int factor

$\hookrightarrow y' + P(x)y = Q(x)$

$y' + P(x)y = Q(x)y^n$ Bernoulli

$f(x)\,dx = g(y)\,dy$ separable

- $e^{x^2+c} = e^c e^{x^2} = ke^{x^2}$

1. exact $\left(\frac{1}{x}+2x\sin y+\frac{1}{y}\right)dx+\left(x^2\cos y-\frac{x}{y^2}\right)dy=$

M N

$M_y = 2x\cos y - \frac{1}{y^2}$

$N_x = 2x\cos y - \frac{1}{y^2}$ $y(2)=\pi$

Soln is $f(x,y)=c$

$f = \int M\,dx = \int N\,dy$

$= \int \frac{1}{x}+2x\sin y+\frac{1}{y}\,dx = \int x^2\cos y - \frac{x}{y^2}\,dy$

$= \underline{\ln x} + x^2\sin y + \frac{x}{y} + A(y) = x^2\sin y + \frac{x}{y} + \underline{B(x)}$

so

$\ln x + x^2\sin y + \frac{x}{y} = c$

IC $\ln 2 + 4\sin\pi + \frac{2}{\pi} = c$

so $\ln x + x^2\sin y + \frac{x}{y} = \ln 2 + \frac{2}{\pi}$

2. homog $(xy+2y^2)\,dx + y^2\,dy = 0$

$y = ux \quad dy = u\,dx + x\,du$

$(ux^2+2u^2x^2)\,dx + u^2x^2(u\,dx + x\,du) = 0$

$dx\,(ux^2+2u^2x^2+u^3x^2) + du\,(u^2x^3) = 0$

$x^2(u+2u^2+u^3)\,dx = -u^2x^3\,du$

$-\frac{1}{x}\,dx = \frac{u^2}{u(1+u)^2}\,du = \frac{u}{(1+u)^2}\,du$

$= \frac{u+1-1}{(1+u)^2}\,du$

$= \frac{1}{1+u} - \frac{1}{(1+u)^2}\,du$

$-\ln|x| + c = \ln|1+u| + \frac{1}{1+u}$ $\quad -\int \frac{1}{w^2}\,dw$

$-\ln|x| + c = \ln\left|1+\frac{y}{x}\right| + \frac{1}{1+y/x}$ $\quad w = 1+u$

3. $(x^3+1)y' - x^2 y = (x^3+1)^{4/3}$

$$y' - \frac{x^2}{x^3+1} y = (x^3+1)^{1/3}$$

$$P = \frac{-x^2}{x^3+1} \qquad \int P\,dx = -\tfrac{1}{3}\ln u = -\tfrac{1}{3}\ln(x^3+1)$$

$$= \ln (x^3+1)^{-1/3}$$

$$u = x^3+1$$

$$k = e^{\int P\,dx} = e^{\ln(x^3+1)^{-1/3}} = (x^3+1)^{-1/3}$$

$$\frac{d}{dx}\left((x^3+1)^{-1/3} y \right) = 1$$

$$(x^3+1)^{-1/3} y = x + c$$

$$y = (x+c)(x^3+1)^{1/3}$$

4. $xy' - y = x^2 y^2$ Bernoulli

$$u = y^{1-n} = y^{-1}$$

$$u' = -y^{-2} y'$$

$$(-y^{-2})xy' - (-y^{-2})y = (-y^{-2})x^2$$

$$xu' + u = -x^2$$

$$u' + \frac{1}{x}u = -x$$

$$P = \frac{1}{x} \qquad k = e^{\int \frac{1}{x}dx} = x$$

$$\frac{d}{dx}(xu) = -x^2$$

$$xu = -\tfrac{1}{3}x^3 + c$$

$$u = -\frac{x^2}{3} + \frac{c}{x}$$

$$y = \frac{1}{-\frac{x^2}{3} + \frac{c}{x}}$$

Bernoulli ? no : $y^2 y^1$

5. $(x^3y^2 + x^3)\frac{dy}{dx} = y$

$x^3(y^2+1)\ dy = y\ dx$

$\frac{y^2+1}{y}\ dy = \frac{1}{x^3}\ dx$

$y + \frac{1}{y}\ dy = x^{-3}\ dx$

$\frac{y^2}{2} + \ln y = -\frac{1}{2}x^{-2} + c$

6. $(\cos x)\,y' = \cos^2 x - y\sin x + 1$

$(\cos x)\,y' + (\sin x)\,y = \cos^2 x + 1$

$y' + (\tan x)\,y = \cos x + \sec x$

$p = \tan x \qquad \int p\,dx = \ln(\sec x) \qquad k = \sec x$

$\frac{d}{dx}(\sec x \cdot y) = \sec x(\cos x + \sec x) = 1 + \sec^2 x$

$\sec x \cdot y = x + \tan x + c$

$y = x\cos x + \sin x + c \cdot \cos x$

SECOND ORDER - 3 questions

7. $y'' + 4y' + 13y = 0$

$y = e^{mx}$ $\quad m^2 + 4m + 13 = 0$

$m = \dfrac{-4 \pm \sqrt{16 - 52}}{2} = \dfrac{-4 \pm 6i}{2}$

$= -2 \pm 3i$

$y = c_1 e^{-2x} \cos 3x + c_2 e^{-2x} \sin 3x$

8. $x^2 y'' + 5xy' + 13y = 0$

$y = x^m$ $\quad m(m-1) + 5m + 13 = 0$

$m^2 + 4m + 13 = 0 \longrightarrow m = -2 \pm 3i$

$y = c_1 x^{-2} \cos(3 \ln x) + c_2 x^{-2} \sin(3 \ln x)$

- Und Coef

$ay'' + by' + cy \equiv Ly = f$

standard forms

$Ly = x^2 - 1 \longrightarrow y_p = Ax^2 + Bx + C$

$Ly = 7e^{-2x}$ $\quad y_p = Ae^{-2x}$

$Ly = 5 \sin 3x$ $\quad y_p = A \sin 3x + B \cos 3x$

duplication

9. $y'' + y = 17 \cos x$

$m^2 + 1 = 0 \quad m = \pm i$ $\quad y_1 = \cos x \quad y_2 = \sin x$

standard $\quad y_p = A \sin x + B \cos x$

dupl so $\quad y_p = Ax \sin x + Bx \cos x$

$= x[\text{standard}]$

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10. $y'' - 2y' + y = 4xe^x$

standard $y_p = (Ax+B)e^x$

$(m-1)^2 = 0 \quad y_1 = e^x, \; y_2 = xe^x$

duplication $Axe^x \leftrightarrow y_2 \quad Be^x \leftrightarrow y_1$

if try $y_p = x[\text{standard}] = Ax^2e^x + Bxe^x$

$\llcorner$ still $\leftrightarrow y_2$

so $y_p = x^2[\quad]$

$= Ax^3e^x + Bx^2e^x$

$y_p' = (3Ax^2e^x + Ax^3e^x) + (2Bxe^x + Bx^2e^x)$

$y_p'' = (6Axe^x + 3Ax^2e^x) + (3Ax^2e^x + Ax^3e^x) + (2Be^x + 2Bxe^x) + (2Bxe^x + Bx^2e^x)$

$[Ax^3e^x + 6Ax^2e^x + 6Axe^x + Bx^2e^x + 4Bxe^x + 2Be^x]$

$-2[Ax^3e^x + 3Ax^2e^x + Bx^2e^x + 2Bxe^x]$

$+[Ax^3e^x + Bx^2e^x]$

$=$

$x^3e^x(A - 2A + A) \qquad\qquad 0x^3e^x \quad \checkmark$

$+x^2e^x(6A + B - 6A - 2B + B) \quad = \quad +0x^2e^x \quad \checkmark$

$+xe^x(6A + 4B - 4B) \qquad\qquad 4xe^x$

$+e^x(2B) \qquad\qquad +0e^x$

$B = 0$

$A = \frac{2}{3}$

$y_p = c_1e^x + c_2xe^x + \frac{2}{3}x^3e^x$

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Variation of Parameters

11 $y'' - 5y' + 4y = e^{7x}$

$m^2 - 5m + 4 = (m-1)(m-4) = 0$

$y_1 = e^x \quad y_2 = e^{4x}$

$W = \begin{vmatrix} e^x & e^{4x} \\ e^x & 4e^{4x} \end{vmatrix} = 4e^{5x} - e^{5x} = 3e^{5x}$

$u_1' = -\frac{f y_2}{W} = -\frac{e^{7x} e^{4x}}{3e^{5x}} = -\frac{1}{3} e^{6x} \qquad u_1 = -\frac{1}{18} e^{6x}$

$u_2' = +\frac{f y_1}{W} = \frac{e^{7x} e^{x}}{3e^{5x}} = \frac{1}{3} e^{3x} \qquad u_2 = \frac{1}{9} e^{3x}$

$y = c_1 e^x + c_2 e^{4x} + \left(-\frac{1}{18} e^{6x}\right)(e^x) + \left(\frac{1}{9} e^{3x}\right)(e^{4x})$

$\quad = \quad " \quad + \quad " \quad + \frac{1}{18} e^{7x}$

2x2 - 1 Questions

12 $x' = x + 3y$ $\quad x(0) = 3$

$y' = -3x - 5y$ $\quad y(0) = 1$

$$\overline{X}' = \begin{bmatrix} 1 & 3 \\ -3 & -5 \end{bmatrix} \overline{X}$$

- $A - \lambda I = \begin{pmatrix} 1-\lambda & 3 \\ -3 & -5-\lambda \end{pmatrix}$

$\det = 0 = (1-\lambda)(-5-\lambda) + 9 = -5 + 5\lambda - \lambda + \lambda^2 + 9$

$= \lambda^2 + 4\lambda + 4 = (\lambda + 2)^2$

$\lambda = -2 \qquad (A - \lambda I)\overline{k} = \overline{0}$

$$\begin{pmatrix} 3 & 3 \\ -3 & -3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \overline{k} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$(A - \lambda I)\overline{P} = \overline{k}$

$$\begin{pmatrix} 3 & 3 \\ -3 & -3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad 3a + 3b = 1$$

let $b = 0$, $a = 1/3$

$$\overline{X}(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t} + c_2 \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} t e^{-2t} + \begin{pmatrix} 1/3 \\ 0 \end{pmatrix} e^{-2t} \right]$$

$$\begin{pmatrix} 3 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1/3 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1/3 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$\det = 1/3$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{1/3}\begin{pmatrix} 0 & -1/3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = 3\begin{pmatrix} -1/3 \\ 4 \end{pmatrix}$$

$c_1 = -1$

$c_2 = 12$

$$\overline{X} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-2t} + 12 \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} t e^{-2t} + \begin{pmatrix} 1/3 \\ 0 \end{pmatrix} e^{-2t} \right]$$

13 $sX(s) - 3 = X(s) + 3Y(s)$

$sY(s) - 1 = -3X(s) - 5Y(s)$

$$\begin{bmatrix} s-1 & -3 \\ 3 & s+5 \end{bmatrix} \begin{bmatrix} X(s) \\ Y(s) \end{bmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\det = (s-1)(s+5) + 9 = s^2 + 4s - 5 + 9$$

$$= (s+2)^2$$

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \frac{1}{(s+2)^2} \begin{pmatrix} s+5 & 3 \\ -3 & s-1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \frac{1}{(s+2)^2} \begin{pmatrix} 3s+18 \\ s-10 \end{pmatrix}$$

$$X(s) = \frac{3s+18}{(s+2)^2} = \frac{3(s+2-2)+18}{(s+2)^2} = \frac{3}{s+2} + \frac{12}{(s+2)^2}$$

$$x(t) = 3e^{-2t} + 12te^{-2t}$$

$$Y(s) = \frac{s-10}{(s+2)^2} = \frac{s+2-12}{(s+2)^2} = \frac{1}{s+2} - \frac{12}{(s+2)^2}$$

$$y(t) = e^{-2t} - 12te^{-2t}$$