4.4 Example 2

$$y'' + 3y = x\cos x - 2\sin x$$

i) $m^2 + 3 = 0 \quad m = \pm\sqrt{3}\,i \qquad y_1 = \sin\sqrt{3}x,\ y_2 = \cos\sqrt{3}x$

ii) standard form $\quad y_p = (Ax+B)\cos x + (Cx+D)\sin x$

iii) no duplication between y_p and y_1 or y_2 so use standard form

$$y_p' = A\cos x - (Ax+B)\sin x + C\sin x + (Cx+D)\cos x$$

$$y_p'' = -A\overset{\sin x}{\cancel{\cos x}} - A\sin x - (Ax+B)\cos x + C\cos x + C\cos x - (Cx+D)\overset{\sin x}{\cancel{\cos x}}$$

$$= -2A\sin x - (Ax+B)\cos x + 2C\cos x - (Cx+D)\overset{\sin x}{\cancel{\cos x}}$$

sub into ODE : $\quad y'' + 3y = \ldots \quad$ categorize into bins

$$\begin{aligned} &\sin x\,[-2A - D + 3D] && -2\sin x \\ &+ \cos x\,[-B + 2C + 3B] \quad = && +0\cos x \\ &+ x\sin x\,[-C + 3C] && +0\,x\sin x \\ &+ x\cos x\,[-A + 3A] && +1\,x\cos x \end{aligned}$$

$$\begin{aligned} -2A + 2D &= -2 \\ 2B + 2C &= 0 \\ 2C &= 0 \\ 2A &= 1 \end{aligned}$$

(3) $B = 0$

(2) $C = 0$

(1) $A = 1/2$

(4) $-1 + 2D = -2$

$D = -1/2$

$$y_p = \left(\tfrac{1}{2}x + 0\right)\cos x + \left(0x - \tfrac{1}{2}\right)\sin x$$

$$y(x) = k_1 \sin\sqrt{3}x + k_2\cos\sqrt{3}x + \tfrac{1}{2}x\cos x - \tfrac{1}{2}\sin x$$

4.4 Example 3

$y'' + 4y' + 3y = e^{-x}$

i) $m^2 + 4m + 3 = 0 \qquad m = -1, -3 \qquad y_1 = e^{-x} \qquad y_2 = e^{-3x}$

ii) standard form for $y_p = Ae^{-x}$

{ this won't work, since $[+Ae^{-x}] + 4[-Ae^{-x}] + 3[Ae^{-x}] \equiv 0$, this means the e^{-x} is the residual of a more complicated y_p; the theory says to multiply the standard form by the lowest power of x that avoids the duplication }

modified form $\quad y_p = x[Ae^{-x}]$

$$y_p' = Ae^{-x} - Axe^{-x}$$

$$y_p'' = -Ae^{-x} - Ae^{-x} + Axe^{-x} = Axe^{-x} - 2Ae^{-x}$$

$[Axe^{-x} - 2Ae^{-x}] + 4[Ae^{-x} - Axe^{-x}] + 3[Axe^{-x}] = \quad \ldots$ two bins

$$\begin{aligned} &e^{-x}[-2A + 4A] \\ +\ &xe^{-x}\underbrace{[A - 4A + 3A]}_{=0} \end{aligned} = \begin{aligned} &1e^{-x} \\ &+0\,xe^{-x} \end{aligned}$$

0 - always occurs in these duplication problems

$2A = 1 \qquad A = \frac{1}{2}$

$y = c_1e^{-x} + c_2e^{-3x} + \frac{1}{2}xe^{-x}$

now include $\quad y(0) = 0, \ y'(0) = 2$

$0 = c_1 + c_2 + 0 \qquad y' = -c_1e^{-x} - 3c_2e^{-3x} + \frac{1}{2}e^{-x} - \frac{1}{2}xe^{-x}$

$\qquad 2 = -c_1 - 3c_2 + \frac{1}{2} - 0$

$c_1 = -c_2 \longrightarrow \quad \frac{3}{2} = +c_2 - 3c_2 = -2c_2$

$c_2 = -\frac{3}{4}$

$c_1 = +\frac{3}{4}$

$y(x) = \frac{3}{4}e^{-x} - \frac{3}{4}e^{-3x} + \frac{1}{2}xe^{-x}$

4.4 Example 4

$$y'' - y' - 2y = e^{2x} - 3e^{-x}$$

$\Downarrow$ $m = -1, 2$ $\quad y_1 = e^{-x} \quad y_2 = e^{2x}$

standard $y_p = \underbrace{Ae^{2x}}_{y_{p_1}} + \underbrace{Be^{-x}}_{y_{p_2}}$

- y_{p_1} duplicates y_2 so modify to $x\left[Ae^{2x}\right]$
- y_{p_2} duplicates y_1 " " " $x\left[Ae^{-x}\right]$

use $y_p = Axe^{2x} + Bxe^{-x}$

$y(x) = c_1 e^{-x} + c_2 e^{2x} + \frac{1}{3} xe^{2x} + xe^{-x}$

Example 5

$$y''' - 2y'' + y' = 7 + (5-2x)e^{x}$$

$m^3 - 2m^2 + m = 0 \quad m(m^2 - 2m + 1) = 0 \quad m(m-1)^2 = 0$

$m = 0, 1, 1$

$y_1 = 1 \quad y_2 = e^{x} \quad y_2 = xe^{x}$

standard $y_p = \underbrace{A}_{y_{p_1}} + \underbrace{(Bx + c)e^{x}}_{y_{p_2}}$

y_{p_1} duplicates y_1, so modify it to $x[A]$

y_{p_2} duplicates y_2, y_3, so try $x[Bx + c]e^{x} \rightarrow (Bx^2 + \underline{Cx)e^{x}}$

still duplicates y_3, so

try $x^2[Bx + c]e^{x} \rightarrow (Bx^3 + Cx^2)e^{x}$

now there's no dupl.

use $y_p = Ax + Bx^3 e^{x} + Cx^2 e^{x}$

result $y = c_1 + c_2 e^{x} + c_3 xe^{x} + 7x - 21x^2 e^{x} + 2x^3 e^{x}$