Test Total

Name ______________________________

Exam 3 — Ordinary Differential Equations — Dr. Kreider

25 Nov 2013 — For full credit, show your work and use correct notation

1. Solve the initial value problem $y' - 6y = 4$, $y(0) = 5$ using Laplace Transforms.

15 pts

$$sY(s) - 5 - 6Y(s) = \frac{4}{s}$$

$$(s-6)Y(s) = 5 + \frac{4}{s}$$

$$Y(s) = \frac{5}{s-6} + \frac{4}{s(s-6)}$$

$$\frac{A}{s} + \frac{B}{s-6}$$

$$4 = A(s-6) + Bs$$

$$s=6 \quad 4 = 6B \quad B = 2/3$$

$$s=0 \quad 4 = A(-6) \quad A = -2/3$$

$$Y(s) = \frac{5}{s-6} - \frac{2/3}{s} + \frac{2/3}{s-6}$$

$$y(t) = 5e^{6t} - \frac{2}{3} + \frac{2}{3}e^{6t}$$

$$= \frac{17}{3}e^{6t} - \frac{2}{3}$$

2. Solve the initial value problem $y'' + 4y' + 40y = 0$, $y(0) = 1, y'(0) = 2$ using Laplace Transforms.

15 pts

$$\left[s^2 Y(s) - s - 2\right] + 4\left[sY(s) - 1\right] + 40Y(s) = 0$$

$$(s^2 + 4s + 40)Y(s) = s + 6$$

$$Y(s) = \frac{s+6}{(s+2)^2 + 36} = \frac{s+2}{(s+2)^2+36} + \frac{4}{(s+2)^2+36} \cdot \frac{6}{6}$$

$$y(t) = e^{-2t}\cos 6t + \frac{2}{3} e^{-2t} \cdot \sin 6t$$

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3. Solve the initial value problem $y'' + 4y = \sin(2t)$, $y(0) = 4, y'(0) = -3$ using Laplace Transforms.

15 pts

$$\left[s^2 Y(s) - 4s + 3\right] + 4Y(s) = \frac{2}{s^2+4}$$

$$(s^2+4)Y(s) = 4s - 3 + \frac{2}{s^2+4}$$

$$Y(s) = \frac{4s-3}{s^2+4} + \frac{2}{(s^2+4)^2} \cdot \frac{16}{16}$$

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$$y(t) = 4\cos 2t - \frac{3}{2}\sin 2t + \frac{1}{8}\left[\sin 2t - 2t\cos 2t\right]$$

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4. Solve the initial value problem $y' - 2y = 2 + 3U(t-6)$, $y(0) = 5$ using Laplace Transforms.

20 pts

$$[sY(s) - 5] - 2Y(s) = \frac{2}{s} + \frac{3}{s}e^{-6s}$$

$$(s-2)Y(s) = 5 + \frac{2}{s} + \frac{3}{s}e^{-6s}$$

$$Y(s) = \frac{5}{s-2} + \underbrace{\frac{2}{s(s-2)} + \frac{3}{s(s-2)}}_{\text{Same PF}} e^{-6s}$$

$$\frac{1}{s(s-2)} = \frac{A}{s} + \frac{B}{s-2}$$

$$1 = A(s-2) + Bs$$

$s=2$ $\quad 1 = 0 + 2B \quad B = 1/2$

$s=0$ $\quad 1 = -2A + 0 \quad A = -1/2$

$$\frac{1}{s(s-2)} = \frac{-1/2}{s} + \frac{1/2}{s-2}$$

$$Y(s) = \frac{5}{s-2} + \underbrace{\left[-\frac{1}{s} + \frac{1}{s-2}\right]}_{\times 2} + \underbrace{\left[-\frac{3/2}{s} + \frac{3/2}{s-2}\right]}_{\times 3} e^{-6s}$$

$$y(t) = 5e^{2t} - 1 + e^{2t} + \left[-\frac{3}{2} + \frac{3}{2}e^{2(t-6)}\right]\mathcal{U}(t-6)$$

5. Solve the initial value problem $y'' - 3y' - 4y = 5\delta(t-8)$, $y(0) = 10, y'(0) = 0$ using Laplace Transforms.

20 pts

$$[s^2 Y(s) - 10s - 0] - 3[sY(s) - 10] - 4Y(s) = 5e^{-8s}$$

$$(s^2 - 3s - 4)Y(s) = 10s - 30 + 5e^{-8s}$$

$$(s-4)(s+1)$$

$$Y(s) = \underbrace{\frac{10s-30}{(s-4)(s+1)}}_{(a)} + \underbrace{\frac{5}{(s-4)(s+1)}}_{(b)} e^{-8s}$$

a) $= \frac{A}{s-4} + \frac{B}{s+1}$ so $10s - 30 = A(s+1) + B(s-4)$

$s=4$ $\quad 10 = 5A + 0 \quad A = 2$

$s=-1$ $\quad -40 = 0 - 5B \quad B = 8$

$$= \frac{2}{s-4} + \frac{8}{s+1}$$

b) $= \frac{A}{s-4} + \frac{B}{s+1}$ so $5 = A(s+1) + B(s-4)$

$s=4$ $\quad 5 = 5A + 0 \quad A = 1$

$s=-1$ $\quad 5 = 0 - 5B \quad B = -1$

$$= \frac{1}{s-4} - \frac{1}{s+1}$$

$$y(t) = \underbrace{\left[2e^{4t} + 8e^{-t}\right]}_{(a)} + \underbrace{\left[e^{4(t-8)} - e^{-(t-8)}\right]}_{(b)} u(t-8)$$

6. Solve the initial value problem using Laplace Transforms:

$$\begin{aligned} x'(t) &= \ 4x(t) + 5y(t) \quad x(0) = 1 \\ y'(t) &= -x(t) + 6y(t) \quad y(0) = 5 \end{aligned}$$

For reference, the inverse of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

15 pts

$$sX(s) - 1 = 4X(s) + 5Y(s)$$
$$sY(s) - 5 = -X(s) + 6Y(s)$$

$$\begin{bmatrix} s-4 & -5 \\ 1 & s-6 \end{bmatrix}\begin{bmatrix} X(s) \\ Y(s) \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

$$\det = (s-4)(s-6) + 5 = s^2 - 10s + 24 + 5 = s^2 - 10s + 29$$
$$= (s-5)^2 + 4$$

$$\begin{bmatrix} X(s) \\ Y(s) \end{bmatrix} = \frac{1}{(s-5)^2+4}\begin{bmatrix} s-6 & 5 \\ -1 & s-4 \end{bmatrix}\begin{bmatrix} 1 \\ 5 \end{bmatrix} = \frac{1}{\det}\begin{bmatrix} s-6+25 \\ -1+5s-20 \end{bmatrix}$$

$$X(s) = \frac{s+19}{(s-5)^2+4} = \frac{s-5 \quad + 24 \cdot \frac{2}{2}}{(s-5)^2+4}$$

$$\bullet \quad x(t) = e^{5t}\cos 2t + 12e^{5t}\sin 2t$$

$$Y(s) = \frac{5s-21}{(s-5)^2+4} = \frac{5(s-5+5)-21}{\quad} = \frac{5(s-5) + 4 \cdot \frac{2}{2}}{(s-5)^2+4}$$

$$\bullet \quad y(t) = 5e^{5t}\cos 2t + 2e^{5t}\sin 2t$$