3450:335 Ordinary Differential Equations, Kreider

1. Below are the equations of motion of various spring mass systems. Interpret the type of damping, and if appropriate, identify the transient and steady state components, and identify if resonance occurs.

   (a) $x(t) = 5te^{-3t} - 6e^{-3t}$

   critical damping

   (b) $x(t) = \frac{1}{13}e^{-3t} + \frac{1}{12}e^{-2t}$

   over damping

   (c) $x(t) = e^{-2t}\sin 4t + 6e^{-2t}\cos 4t$

   underdamping

   (d) $x(t) = \underbrace{5te^{-3t} - 6e^{-3t}}_{\text{critical damping (transient)}} + \underbrace{2\sin 4t}_{\text{plus an external source (steady state)}}$

   (e) $x(t) = \underbrace{2e^{-t}\sin t + 5e^{-t}\cos t}_{\text{underdamping (transient)}} + \underbrace{6\cos 4t}_{\text{plus external source (steady state)}}$

   (f) $x(t) = \underbrace{2\sin 4t + 5\cos 4t}_{\text{SHM, no damping}} + 6t\cos t$

   resonance

2. Set up but do not solve. A 5 kg object stretches a spring 2 m. The object is replaced by a 2 kg mass, and the spring-mass system is placed in a liquid with damping factor 12. The mass is released from the point 1 m below equilibrium with initial downward velocity 2 m/s. Set up the initial value problem for the equation of motion, but do not solve.

$ks = mg$, $k \cdot 2 = 5 \cdot 9.8$, $k = 24.5$

$2x'' + 12x' + 24.5x = 0$

$x(0) = +1$

$x'(0) = +2$

3. Solve $y'' + 5y' + 4y = e^{-4t} + 3e^{t}$ by undetermined coefficients.

1) $m^2 + 5m + 4 = 0$ $(m+4)(m+1) = 0$ $m = -4, -1$ $y_1 = e^{-4t}$

$y_2 = e^{-t}$

2) standard $y_p = \underline{Ae^{-4t}} + \underline{Be^{t}}$

duplication with y_1 no duplication

3) modify: $y_p = Ate^{-4t} + Be^{t}$

$y_p' = Ae^{-4t} - 4Ate^{-4t} + Be^{t}$

$y_p'' = -4Ae^{-4t} - 4Ae^{-4t} + 16Ate^{-4t} + Be^{t}$

4) $[-8Ae^{-4t} + 16Ate^{-4t} + Be^{t}]$

$+5[Ae^{-4t} - 4Ate^{-4t} + Be^{t}]$

$+4[Ate^{-4t} + Be^{t}]$

$= \underline{-3Ae^{-4t}} + 0 + \underline{\underline{10Be^{t}}} = \underline{e^{-4t}} + \underline{\underline{3e^{t}}}$

$A = -\frac{1}{3}$ $B = \frac{3}{10}$

$y = c_1 e^{-4t} + c_2 e^{-t} - \frac{1}{3}\underline{te^{-4t}} + \frac{3}{10}e^{t}$

4. Solve $y'' + 15y = \sin 15t$ by undetermined coefficients.

$m^2 + 15 = 0 \quad m = \pm\sqrt{15}\, i \quad y_1 = \sin(\sqrt{15}t) \quad y_2 = \cos(\sqrt{15}t)$

Standard $y_p = A\sin 15t + B\cos 15t$: no duplication

$y_p' = 15A\cos 15t - 15B\sin 15t$

$y_p'' = -225A\sin 15t - 225B\cos 15t$

$$[-225A\,S - 225B\,C] + 15[A\,S + B\,C] = S$$

$\sin 15t$: $-225A + 15A = 1 \qquad A = \frac{1}{210}$

$\cos 15t$: $-225B + 15B = 0 \qquad B = 0$

$$y = c_1 \sin\sqrt{15}t + c_2\cos\sqrt{15}t + \frac{1}{210}\sin 15t$$

5. Solve $y'' + 15y = \sin\sqrt{15}t$ by undetermined coefficients.

again $y_1 = \sin\sqrt{15}t$, $y_2 = \cos\sqrt{15}t$

now y_p has duplication, so use $y_p = At\sin\sqrt{15}t + Bt\cos\sqrt{15}t$

$y_p' = A\,S + \sqrt{15}At\,C + B\,C - \sqrt{15}\,Bt\,S$

$y_p'' = \sqrt{15}A\,C + [\sqrt{15}A\,C - 15At\,S] - \sqrt{15}B\,S + [-\sqrt{15}B\,S - 15Bt\,C]$

$$[2\sqrt{15}A\,C - 15At\,S - 2\sqrt{15}B\,S - 15Bt\,C]$$
$$+ 15[At\,S + Bt\,C]$$

$$= 2\sqrt{15}A\cos\sqrt{15}t + 0 - 2\sqrt{15}B\sin\sqrt{15}t + 0 = \sin\sqrt{15}t$$

$-2\sqrt{15}B = 1$

$A = 0 \qquad B = -\frac{1}{2\sqrt{15}}$

$$y = c_1\sin\sqrt{15}t + c_2\cos\sqrt{15}t - \frac{1}{2\sqrt{15}}t\cos\sqrt{15}t$$

6. Solve $y'' - 2y' - 3y = 64e^{-x}$ by variation of parameters.

$$m^2 - 2m - 3$$
$$(m-3)(m+1) \qquad y_1 = e^{3x} \qquad y_2 = e^{-x}$$

$$W = \begin{vmatrix} e^{3x} & e^{-x} \\ 3e^{3x} & -e^{-x} \end{vmatrix} = -e^{2x} - 3e^{2x} = -4e^{2x}$$

$$u_1' = -\frac{fy_2}{W} = \frac{-64e^{-x}e^{-x}}{-4e^{2x}} = 16e^{-4x}$$

$$u_1 = 16\frac{e^{-4x}}{-4} = -4e^{-4x}$$

$$u_2' = +\frac{fy_1}{W} = \frac{64e^{-x} \cdot e^{3x}}{-4e^{2x}} = -16$$

$$u_2 = -16x$$

$$y = c_1 e^{3x} + c_2 e^{-x} + (-4e^{-4x})(e^{3x}) + (-16x)(e^{-x})$$
$$= c_1 e^{3x} + \underbrace{c_2 e^{-x} - 4e^{-x}} - 16xe^{-x}$$
$$= c_1 e^{3x} + d_2 e^{-x} - 16xe^{-x}$$

we can relabel $d_2 = c_2 - 4$

<u>if</u> we want

7. Solve $\bar{X}' = \begin{pmatrix} -4 & -3 \\ 2 & 1 \end{pmatrix} \bar{X}$ using eigenvalues and eigenvectors.

$$A - \lambda I = \begin{pmatrix} -4-\lambda & -3 \\ 2 & 1-\lambda \end{pmatrix}$$

$$\det = (-4-\lambda)(1-\lambda) - (-6) = -4 - \lambda + 4\lambda + \lambda^2 + 6 = \lambda^2 + 3\lambda + 2$$
$$= (\lambda+1)(\lambda+2)$$
$$\lambda = -1, -2$$

$\lambda_1 = -1$ $\begin{pmatrix} -3 & -3 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $2a + 2b = 0$

$a = -b$ use $\bar{K}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\lambda_2 = -2$ $\begin{pmatrix} -2 & -3 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $2a + 3b = 0$ • "cross multiply"

$a = -3 \leftrightarrow b = 2$

$\bar{K}_2 = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$

$$\bar{X} = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -3 \\ 2 \end{pmatrix} e^{-2t}$$

8. Solve $\bar{X}' = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix} \bar{X}$ using eigenvalues and eigenvectors.

$$A - \lambda I = \begin{pmatrix} 3-\lambda & 1 \\ -1 & 1-\lambda \end{pmatrix}$$

$$\det = (3-\lambda)(1-\lambda) - (-1) = 3 - \lambda - 3\lambda + \lambda^2 + 1 = \lambda^2 - 4\lambda + 4$$
$$= (\lambda - 2)^2$$
$$\lambda = 2, 2$$

$(A - \lambda I)\bar{K} = \bar{0}$ $\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $a + b = 0$

$a = -b$ use $\bar{K} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$(A - \lambda I)\bar{P} = \bar{K}$ $\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $a + b = -1$

if $b = 0$ then $a = -1$

use $\bar{P} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$$\bar{X} = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{2t} + c_2 \left[\begin{pmatrix} -1 \\ 1 \end{pmatrix} t e^{2t} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} e^{2t} \right]$$