Applied Numerical Methods I 3450:427/527, Kreider
Introduction to Rootfinding

The history of root finding goes back a long way. In 1225, Leonardo of Pisa studied the equation $x^3 + 2x^2 + 10x - 20 = 0$ and found a root $x = 1.368808107$. No one knows what method he used, but it's a pretty impressive achievement.

There are many equations that admit exact solutions; for example, $ax+b = 0$, $ax^2+bx+c = 0$, $\sin x = 0$. However, most equations are nonlinear, and cannot be solved exactly. We are able to obtain approximate solutions as accurately as we want with relative ease in most cases, however.

a. There are 2 basic forms of nonlinear equation: $f(x) = 0$ and $g(x) = x$. A solution to the first form is called a root, and a solution to the second form is called a fixed point. Any equation can be put into either form. As an example, consider $\sin(x) = (x/2)^2$. This can be written as $f(x) = \sin(x) - (x/2)^2 = 0$. Or you can add x to both sides of f to get $x = g(x) = x + \sin(x) - (x/2)^2$. Or you can solve the original equation for either of the x's that appear: $x = \arcsin(x/2)^2$ or $2\sqrt{\sin x} = x$.

   Systems of equations can be studied as well. The notation is $\bar{f}(\bar{x}) = \bar{0}$, where the bar indicates a vector. An example will be given on the board in class (so I don't have to type it).

b. Difficulties. There are several problems that may occur when trying to find an approximate solution to a nonlinear equation.

   - The number of solutions varies, and may be unknown for a given equation. For example, $e^x + 1 = 0$ has no solutions, $e^{-x} - x = 0$ has one solution, $x^2 - 4\sin x = 0$ has 2 solutions, $x^3 + 6x^2 + 11x - 6 = 0$ has three solutions, and $\sin x = 0$ has infinitely many solutions.
   - Roots may be too close to resolve. For example, $0 = 3x^2 + \frac{1}{\pi^4}\ln(\pi - x)^2 + 1$ has roots $x = \pi \pm 10^{-667}$. These are VERY close together.
   - You may not know where to look for a root, and might therefore miss it completely. For example, $0 = x^5 + 9875x^4 - 9876x^3 + x^2 + 9875x - 9876$ has the graph shown on the board in class.
   - The root may be a multiple root ($f(x) = 0$ and $f'(x) = 0$ at the root). This can lead to computational problems that will be discussed in the coming lectures.

c. There are 2 types of methods: (i) bracketing methods isolate a root in an interval and then zoom in to resolve it, and (ii) shooting methods start with an initial guess and then refine it. Both types are *iterative*; that is, we construct a sequence of approximations x_n that converges to the root c.

d. Natural Questions

   - How do we know where to start?

     Partial answer: computer graphics, a table of function values, divine intervention ... The main point is that you need to know *something* about the location of any roots before proceeding.

- How many iterations are needed?

  Partial answer: It depends . . . on the method, the initial guess and the desired precision. The speed of an algorithm is measured by its *rate of convergence*. Suppose that $|x_n - c| \leq a_n$. The convergence is said to be

  - linear if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \alpha$, where $0 < \alpha < 1$.
  - sublinear if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$
  - of order p if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n^p}\right| = \alpha$, where $0 < \alpha < \infty$.

  If $p = 2$, the method is quadratic, if $p = 3$ the method is cubic, etc, and if $1 < p < 2$, the method is superlinear. The bigger p is, the faster the method is (the fewer iterations are needed to obtain a given precision).

- Will the method always work?

  Partial answer: It depends on the method.