3450:427/527 Applied Numerical Methods I, Kreider

Order of Convergence Examples

Here's the thought process behind the order of convergence. Let $x_k \to c$, with error $\epsilon_k = |x_k - c|$. If the order of convergence is p, then the ratio $\frac{\epsilon_{k+1}}{\epsilon_k^p}$ is balanced; that is, $\epsilon_{k+1} \approx \beta \epsilon_k^p$. If you consider $\frac{\epsilon_{k+1}}{\epsilon_k^r}$ where r is smaller than p, the ratio goes to 0 because the denominator is BIGGER than the numerator. If you consider $\frac{\epsilon_{k+1}}{\epsilon_k^r}$ where r is larger than p, the ratio goes to infinity because the denominator is SMALLER than the numerator. Let's look at this idea for 2 root finding methods.

The fixed point method converges linearly, while Newton's Method converges quadratically. To see what this means in terms of the convergence order definitions, let's compute the appropriate ratios for a set of problems. In the first problem, we are finding the solution to $\tan(x) = 5x$ by the fixed point method, starting with $x_1 = 1$. Clearly, the ratio $\dfrac{\epsilon_{k+1}}{\epsilon_k}$ is going to a constant less than 1, indicating linear convergence. In the second problem, we are computing $\sqrt{2}$ using Newton's Method.

Fixed Point Method

k	$\dfrac{\epsilon_{k+1}}{\epsilon_k}$	$\dfrac{\epsilon_{k+1}}{\epsilon_k^2}$	$\dfrac{\epsilon_{k+1}}{\epsilon_k^3}$
1	1.35711e-001	3.14122e-001	7.27080e-001
2	9.96623e-002	1.69981e+000	2.89914e+001
3	9.60454e-002	1.64367e+001	2.81289e+003
4	9.56978e-002	1.70515e+002	3.03826e+005
5	9.56645e-002	1.78119e+003	3.31643e+007
6	9.56613e-002	1.86185e+004	3.62372e+009
7	9.56610e-002	1.94629e+005	3.95986e+011
8	9.56610e-002	2.03457e+006	4.32723e+013
9	9.56610e-002	2.12685e+007	4.72868e+015
10	9.56612e-002	2.22333e+008	5.16739e+017

Newton's Method

k	$\dfrac{\epsilon_{k+1}}{\epsilon_k}$	$\dfrac{\epsilon_{k+1}}{\epsilon_k^2}$	$\dfrac{\epsilon_{k+1}}{\epsilon_k^3}$
2	2.860E-2	3.333E-1	3.886E+0
3	8.658E-4	3.529E-1	1.439E+2
4	7.508E-7	3.535E-1	1.665E+5

Clearly, $\dfrac{\epsilon_{k+1}}{\epsilon_k} \to 0$, indicating that the convergence is better than linear, while $\dfrac{\epsilon_{k+1}}{\epsilon_k^3} \to \infty$, indicating that convergence is worse than cubic. The ratio $\dfrac{\epsilon_{k+1}}{\epsilon_k^2}$ goes to a constant, indicating the convergence is quadratic.

Note: for some methods, it is possible for the ratio $\dfrac{\epsilon_{k+1}}{\epsilon_k} \to 0$ and the ratio $\dfrac{\epsilon_{k+1}}{\epsilon_k^2} \to \infty$. That indicates that the order of convergence p is between 1 and 2, so $1 < p < 2$.