3450:427/527 Applied Numerical Methods I, Kreider

Notes on Newton's Method

Newton's Method is a form of fixed point iteration. Note the formula:

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ x_{n+1} &= g(x_n) \end{aligned}$$

From this observation, we can gather quite a lot of information.

1. To determine when Newton's Method converges, use the Fixed Point Theorem – we need $|g'(x)| < 1$ to get convergence. Note that

$$\begin{aligned} g'(x) &= 1 - \frac{[f'(x)]f'(x) - f(x)[f''(x)]}{(f'(x))^2} \\ &= \frac{f(x)f''(x)}{(f'(x))^2} \end{aligned}$$

This indicates that Newton's Method converges whenever the iterates x_n are in an interval where $\left|\frac{f(x)f''(x)}{(f'(x))^2}\right| < 1$. This interval is probably not trivial to determine for any given problem, so we usually don't try, but if Newton's Method fails, the problem is usually that this condition is not met.

2. To determine the order of convergence, first factor the root from f by writing

$$f(x) = h(x)(x-c)^M$$

where $h(c) \neq 0$ and M is the multiplicity of the root. As a simple example, $f(x) = x^5 - 15x^4 + 77x^3 - 155x^2 + 150x - 250$ can be written as $f(x) = (x^2+2)(x-5)^3$, so $h(x) = x^2 + 2$ ($h(5) \neq 0$) and $M = 3$. This means that the fixed point function $g(x)$ is

$$\begin{aligned} g(x) &= x - \frac{f(x)}{f'(x)} \\ &= x - \frac{h(x)(x-c)^M}{h'(x)(x-c)^M + Mh(x)(x-c)^{M-1}} \\ &= x - \frac{h(x)(x-c)}{h'(x)(x-c) + Mh(x)} \end{aligned}$$

Now, we need $|g'(x)| < 1$ in an interval containing $x = c$ for convergence, so if g' is continuous and $|g'(c)| < 1$, then such an interval exists. Let's check to see what $g'(x)$ looks like.

$$\begin{aligned} g'(x) &= 1 - \frac{[h'(x)(x-c) + Mh(x)]\{h'(x)(x-c) + h(x)\} - h(x)(x-c)[h''(x)(x-c) + h'(x) + Mh'(x)]}{(h'(x)(x-c) + Mh(x))^2} \\ g'(c) &= 1 - \frac{Mh^2(c)}{M^2h^2(c)} = 1 - \frac{1}{M} \end{aligned}$$

3. To interpret what $g'(c)$ tells us about the order of convergence, consider the error expression

$$\begin{aligned} \epsilon_{n+1} = x_{n+1} - c &= g(x_n) - g(c) \\ &= \left[g(c) + g'(c)(x_n - c) + \frac{1}{2}g''(c)(x_n - c)^2 + \cdots\right] - g(c) \\ &= g'(c)(x_n - c) + \frac{1}{2}g''(c)(x_n - c)^2 + \cdots \end{aligned}$$

Simple Root. When $M = 1$, the expression for $g'(c)$ reduces to 0. It can also be shown that $g''(c)$ is not 0 (evaluating $g''(x)$ is tedious but straightforward). So the error expression looks like

$$\begin{aligned} \epsilon_{n+1} &= \frac{1}{2}g''(c)(x_n - c)^2 + \cdots \qquad \text{so that} \\ \frac{\epsilon_{n+1}}{(x_n - c)^2} &= \frac{\epsilon_{n+1}}{\epsilon_n^2} \to \frac{1}{2}g''(c) = \beta \end{aligned}$$

This means that Newton's Method converges quadratically for a simple root.

Multiple Root. If $M > 1$, then $g'(c) \neq 0$, so we have

$$\frac{\epsilon_{n+1}}{x_n - c} = \frac{\epsilon_{n+1}}{\epsilon_n} \to g'(c) = \alpha$$

and the convergence is now linear. It will take more iterations for Newton's Method to converge to a multiple root.

4. Adjusting Newton's Method to regain quadratic convergence

   To allow for the possibility of a multiple root, we can modify Newton's Method slightly. To goal is to make $g'(c) = 0$, which is the condition for quadratice convergence. If we introduce a constant β in the formula as

$$\begin{aligned} x_{n+1} &= x_n - \frac{\beta f(x_n)}{f'(x_n)} \qquad \text{so that} \\ g(x) &= x - \frac{\beta f(x)}{f'(x)} \end{aligned}$$

   then clearly $g'(c) = 1 - \dfrac{\beta}{M}$. To make $g'(c) = 0$, set $\beta = M$. Hence, the modified iteration equation is

$$x_{n+1} = x_n - \frac{M f(x_n)}{f'(x_n)}$$

   If you know the multiplicity of the root, you can then use that factor to your advantage by using the modified Newton iteration formula. If you don't know the multiplicity (a very common occurence), then you should use the regular formula with $M = 1$ and settle for linear convergence.