3450:427/527 Applied Numerical Methods I, Kreider

Ill Conditioning Example

Here is a simple example of ill conditioning. Suppose that $Ax = b$ is supposed to be

$$\begin{aligned} 2x \quad &+6y &= 8 \\ 2x \quad &+6.00001y &= 8.00001 \end{aligned}$$

The actual solution is $x = 1$, $y = 1$. Suppose further that due to representation error, the system on the machine is changed slightly to

$$\begin{aligned} 2x \quad &+6y &= 8 \\ 2x \quad &+5.99999y &= 8.00002 \end{aligned}$$

The solution to this system is $x = 10$, $y = -2$, so you think the answer is $(10, -2)$. When you check the answer by plugging these values into the actual system, you get

$$\begin{aligned} 2(10) + 6(-2) &= 8 \\ 2(10) + 5.99999(-2) &= 7.99998 \end{aligned}$$

This seems to be acceptable, but of course $(10, -2)$ is very far from the actual solution $(1, 1)$. This indicates that the system is badly ill conditioned.

Here are some things to consider if you have an ill conditioned system:

- To identify if the matrix is ill conditioned, you can try 2 things. First, compute $cond(A)$. This is relatively expensive and sometimes hard to interpret because the value may be in an intermediate range. Second, you can introduce deliberate "representation errors" by slightly perturbing one or more elements in A. Call the new matrix A', and solve $A'x' = b$. If $x \approx x'$, then there is probably no ill conditioning. The danger here is that you might be unlucky, and chose the wrong element to perturb. But if you try this several times with different elements and all the solutions are about the same, then you have confidence that the matrix is well conditioned.

- If the system really is ill conditioned, there is no simple fix. Consider using Singular Value Decomposition (SVD) or a "regularization" technique.