5.5 a $\tan^{-1}x$ and $\sin^{-1}x$

$\int \frac{1}{1+u^2}\,du = \tan^{-1}u + c$ $\quad \int \frac{1}{\sqrt{1-u^2}}\,du = \sin^{-1}u + c$

1. $I = \int \frac{1}{1+4x^2}\,dx$

$\hookrightarrow u^2 = 4x^2$ so $u = 2x$, $du = 2dx$, $dx = \frac{1}{2}du$

$I = \int \frac{1}{1+u^2}\left(\frac{1}{2}du\right) = \frac{1}{2}\tan^{-1}(u) + c$

$= \frac{1}{2}\tan^{-1}(2x) + c$

2. $I = \int \frac{4x}{1+x^4}\,dx$

$\llcorner\ u^2 = x^4$ so $u = x^2$, $du = 2x\,dx$, $x\,dx = \frac{1}{2}du$

$I = \int \frac{4}{1+u^2}\left(\frac{1}{2}du\right) = 2\tan^{-1}(u) + c = 2\tan^{-1}(x^2) + c$

3. $I = \int \frac{\cos x}{1+\sin^2 x}\,dx$

$\llcorner\ u^2 = \sin^2 x$ so $u = \sin x$, $du = \cos x\,dx$

$I = \int \frac{1}{1+u^2}\,du = \tan^{-1}u + c = \tan^{-1}(\sin x) + c$

4. $I = \int \frac{4x^2}{1+x^6}\,dx$

$\llcorner\ u^2 = x^6$ so $u = x^3$, $du = 3x^2\,dx$

$x^2\,dx = \frac{1}{3}du$

$I = \int \frac{4}{1+u^2}\left(\frac{1}{3}du\right)$

$= \frac{4}{3}\tan^{-1}(u) + c$

$= \frac{4}{3}\tan^{-1}(x^3) + c$

5.5 G 2

5. $I = \int \frac{2}{1+3x^2}\,dx$

$\hookrightarrow u^2 = 3x^2 \quad u = \sqrt{3}x, \quad du = \sqrt{3}\,dx, \quad dx = \frac{1}{\sqrt{3}}\,du$

$I = \int \frac{2}{1+u^2}\left(\frac{1}{\sqrt{3}}\,du\right)$

$= \frac{2}{\sqrt{3}}\tan^{-1}(u) + c = \frac{2}{\sqrt{3}}\tan^{-1}(\sqrt{3}x) + c$

6. $I = \int \frac{1}{4+x^2}\,dx$ The template requires a 1, so manipulate the denominator to get a 1 there

$I = \int \frac{1}{4\left[1+\frac{x^2}{4}\right]}\,dx = \frac{1}{4}\int \frac{1}{1+\frac{x^2}{4}}\,dx$

$\hookrightarrow u^2 = \frac{x^2}{4}$

$u = \frac{x}{2} \quad du = \frac{1}{2}\,dx \quad dx = 2\,du$

$I = \frac{1}{4}\int \frac{1}{1+u^2}\left(2\,du\right) = \frac{1}{2}\tan^{-1}(u) + c$

$= \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$

7. $I = \int \frac{1}{3+x^2}\,dx = \frac{1}{3}\int \frac{1}{1+\frac{x^2}{3}}\,dx$

$u^2 = \frac{x^2}{3} \quad u = \frac{x}{\sqrt{3}} \quad dx = \sqrt{3}\,du$

$I = \frac{1}{3}\int \frac{1}{1+u^2}\left(\sqrt{3}\,du\right) = \frac{\sqrt{3}}{3}\tan^{-1}(u) + c = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + c$

GENERAL RULE

$$\int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + c$$

5.5 a 3

8. $I = \int \frac{1}{6+7x^2}\,dx = \frac{1}{7}\int \frac{1}{\frac{6}{7}+x^2}\,dx$

$a^2 = \frac{6}{7}$ so $a = \sqrt{\frac{6}{7}}$

$I = \frac{1}{7}\frac{1}{\sqrt{6/7}}\tan^{-1}\left(\frac{x}{\sqrt{6/7}}\right) + C$ ← from the new template

OR

$I = \frac{1}{6}\int \frac{1}{1+\frac{7x^2}{6}}\,dx$

$u^2 = \frac{7x^2}{6}$ $u = \sqrt{\frac{7}{6}}\,x$ $du = \sqrt{\frac{7}{6}}\,dx$

$dx = \sqrt{\frac{6}{7}}\,dx$

$I = \frac{1}{6}\int \frac{1}{1+u^2}\left(\sqrt{\frac{6}{7}}\,dx\right)$

$= \frac{\sqrt{6}}{6}\frac{1}{\sqrt{7}}\tan^{-1}(u) + c = \frac{1}{\sqrt{42}}\tan^{-1}\left(\sqrt{\frac{7}{6}}\,x\right) + c$

note: $\frac{1}{7\sqrt{6/7}} = \frac{1}{7\frac{1}{\sqrt{7}}\sqrt{6}} = \frac{1}{\sqrt{7}\sqrt{6}} = \frac{1}{\sqrt{42}}$

5.5 Q 4

$$\int \frac{1}{\sqrt{1-u^2}}\,du = \sin^{-1}(u) + c$$

9. $I = \int \frac{x}{\sqrt{1-x^4}}\,dx$

$u^2 = x^4 \quad u = x^2 \quad du = 2x\,dx \quad x\,dx = \frac{1}{2}\,du$

$I = \int \frac{1}{\sqrt{1-u^2}}\left(\frac{1}{2}du\right) = \frac{1}{2}\sin^{-1}(u) + c = \frac{1}{2}\sin^{-1}(x^2) + c$

10. $I = \int \frac{\sec^2 x}{\sqrt{1-\tan^2 x}}\,dx$

$u^2 = \tan^2 x, \quad u = \tan x, \quad du = \sec^2 x\,dx$

$I = \int \frac{1}{\sqrt{1-u^2}}\,du = \sin^{-1}(u) + c = \sin^{-1}(\tan x) + c$

11. $I = \int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx \quad u^2 = e^{2x} \quad u = e^x \quad du = e^x\,dx$

$I = \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1}(u) + c = \sin^{-1}(e^x) + c$

BUT

derivative of e^{2x}

12. $I = \int \frac{e^{2x}}{\sqrt{1-e^{2x}}}\,dx$

so

$u = 1 - e^{2x}$

$du = -2e^{2x}\,dx$

$e^{2x}\,dx = -\frac{1}{2}\,du$

$I = \int \frac{1}{\sqrt{u}}\left(-\frac{1}{2}du\right)$

$= -\frac{1}{2}\int u^{-1/2}\,du$

$= -\frac{1}{2}\,\frac{u^{1/2}}{1/2} + c = -\sqrt{1-e^{2x}} + c$

5.5 G 5

13 $I = \int \frac{1}{\sqrt{1-9x^2}}\,dx$ $\qquad u^2 = 9x^2$

$u = 3x \quad du = 3dx \quad dx = \frac{1}{3}du$

$I = \int \frac{1}{\sqrt{1-u^2}}\left(\frac{1}{3}du\right) = \frac{1}{3}\sin^{-1}(3x) + c$

14 $I = \int \frac{1}{\sqrt{9-x^2}}\,dx$

need a 1 for the template

$= \int \frac{1}{\sqrt{9\left(1-\frac{x^2}{9}\right)}}\,dx = \frac{1}{3}\int \frac{1}{\sqrt{1-\frac{x^2}{9}}}\,dx$

$u^2 = \frac{x^2}{9} \quad u = \frac{x}{3} \quad du = \frac{1}{3}dx$

$dx = 3du$

$I = \frac{1}{3}\int \frac{1}{\sqrt{1-u^2}}(3du)$

$= \sin^{-1}(u) + c = \sin^{-1}\left(\frac{x}{3}\right) + c$

GENERAL RULE

$$\int \frac{1}{\sqrt{a^2-u^2}}\,du = \sin^{-1}\left(\frac{u}{a}\right) + c$$

15 $I = \int \frac{1}{\sqrt{9-4x^2}}\,dx = \int \frac{1}{\sqrt{9\left(1-\frac{4x^2}{9}\right)}}\,dx$

$= \frac{1}{3}\int \frac{1}{\sqrt{1-\frac{4x^2}{9}}}\,dx$ $\qquad u^2 = \frac{4x^2}{9}, \quad u = \frac{2x}{3}$

$du = \frac{2}{3}dx, \quad dx = \frac{3}{2}du$

$I = \frac{1}{3}\int \frac{1}{\sqrt{1-u^2}}\left(\frac{3}{2}du\right) = \frac{1}{2}\sin^{-1}(u) + c$

$= \frac{1}{2}\sin^{-1}\left(\frac{2x}{3}\right) + c$