Section 5.5 part A: Overview

1. We know that

$$\int x^6\,dx = \frac{x^7}{7} + c$$

2. so it is reasonable to assume that

$$\int (x+3)^6\,dx = \frac{(x+3)^7}{7} + c$$

   but is it true? We can check it by differentiating: $\frac{d}{dx}\left(\frac{(x+3)^7}{7} + c\right) = \frac{7(x+3)^6[1]}{7} + 0 = (x+3)^6$. Ok, that's a handy thing to know.

3. So now let's try in on a similar problem. Is it true that

$$\int (2x+3)^6\,dx = \frac{(2x+3)^7}{7} + c?$$

   the answer is no, since differentiation gives $\frac{d}{dx}\left(\frac{(2x+3)^7}{7} + c\right) = \frac{7(2x+3)^6[2]}{7} + 0 = 2(2x+3)^6$. Our reasoning has failed to take the chain rule into account - that's what gives us the 2 in the derivative.

4. Therefore, we need a method to account for chain rule effects. It is called Substitution. We will replace the original variable x with a new variable u in order to make the structure of the integrand more simple.

5. Here is the general approach. For any function $F(x)$, we know that

$$\int \frac{d}{dx}F(x)\,dx = F(x) + c$$

   which means that

$$\int \frac{d}{dx}f(u(x))\,dx = f(u(x)) + c$$

   We also know that $\frac{d}{dx}f(u(x)) = f'(u(x)) \cdot u'(x)$; that's the chain rule. So then

$$\int f'(u(x)) \cdot u'(x)\,dx = f(u(x)) + c$$

   Note also that $u'(x)dx = du$, so we could write this as

$$\int f'(u)\,du = f(u) + c$$

   which is consistent with our standard notation.

6. But what does this give us? The form that is most helpful is

$$\int f'(u(x)) \cdot u'(x)\,dx = f(u(x)) + c$$

The component $f'(u(x))$ is a composite function, so we'll be looking for that. And then the $u'(x)$ component sits outside the composite function, so that's generally easy to identify. Once we identify those 2 components, then we 'complete the substitution' to get the antiderivative. Our goal is to find $u(x)$ so that we can rewrite the integral in a simpler form. The x pieces have to be converted to u, and the dx has to be converted to du.

7. Here's an example. Evaluate $\int 2(2x+3)^6\,dx$. Look for the composite function; it's $(2x+3)^6$. Now, what is outside that? It's $2dx$, and that will form the du. Now that we have that recognition, it's time to identify u. That is what is inside the composite function, so $u = 2x+3$. This means that $du = 2dx$ (remember, to get the differential, you take the derivative and multiply by dx). We have

$$\begin{aligned}
\int 2(2x+3)^6\,dx &= \int (2x+3)^6\,[2dx] \\
&\qquad\qquad u = 2x+3 \\
&\qquad\qquad du = 2dx \\
\int 2(2x+3)^6\,dx &= \int u^6\,du \\
&= \frac{u^7}{7} + c = \frac{(2x+3)^7}{7} + c
\end{aligned}$$

The key idea is that by doing the substitution, we get a recognizable form, $\int u^6\,du$, that we can integrate.

8. The Structure of a Substitution Problem

Here are some templates for us to fill in. The function provided is the composite function (that's the $f'(u(x))$ in the formulas above). We need to identify u (what's inside) and then we need u' outside the composite so that it can be associated with the dx to complete the substitution.

$$\begin{aligned}
I &= \int \underline{\quad\quad} \sqrt{1+x^2}\,dx \\
I &= \int \underline{\quad\quad} \sin x^3\,dx \\
I &= \int \underline{\quad\quad} \frac{1}{1+x^4}\,dx \\
I &= \int \underline{\quad\quad} \cos(e^x+2)\,dx \\
I &= \int \underline{\quad\quad} (5x+4)^{17}\,dx
\end{aligned}$$

So ... for each one, identify u and put u' in the underlined area:

$$
\begin{aligned}
I &= \int \underline{2x}\sqrt{1+x^2}\,dx, \qquad u = 1+x^2 \\
I &= \int \underline{3x^2}\sin x^3\,dx, \qquad u = x^3 \\
I &= \int \underline{4x^3}\frac{1}{1+x^4}\,dx, \qquad u = 1+x^4 \\
I &= \int \underline{e^x}\cos(e^x+2)\,dx, \qquad u = e^x+2 \\
I &= \int \underline{5}(5x+4)^{17}\,dx, \qquad u = 5x+4
\end{aligned}
$$

The next step is to move the underlined part next to the dx so we recognize that $u'dx = du$:

$$
\begin{aligned}
I &= \int \sqrt{1+x^2}\,(2x dx) \\
I &= \int \sin x^3\,(3x^2 dx) \\
I &= \int \frac{1}{1+x^4}\,(4x^3 dx) \\
I &= \int \cos(e^x+2)\,(e^x dx) \\
I &= \int (5x+4)^{17}\,(5dx)
\end{aligned}
$$

Now, complete the substitution by plugging in the u and du:

$$
\begin{aligned}
I &= \int \sqrt{u}\,du \\
I &= \int \sin u\,du \\
I &= \int \frac{1}{u}\,du \\
I &= \int \cos(u)\,du \\
I &= \int u^{17}\,du
\end{aligned}
$$

Each of these forms is now recognizable and easy to integrate. We'll stop here because there are plenty of examples in the subsequent lecture sections. In those sections, I'll write the notes by hand so we can see how it is that we write everything out.