Section 5.4 part C: Accumulation Problems

1. Integrals are used to measure how quantities accumulate. The general statement for accumulation from starting time $t = a$ to ending time t is

$$\text{accumulated quantity} = \int_a^t (\text{rate of accumulation})\, dt$$

If we label the quantity as P, then the general statement looks like

$$P = \int_a^t \frac{dP}{dt}\, dt$$

2. Example 1. Rain falls at the rate $\frac{1}{t^2}$ inches per hour (in/hr) starting at 1:00pm (we'll label this as $a = 1$). How much rain has fallen by 1:30pm, by 3:00pm, by 6:00pm? Converting times to t values, that's $t = 1.5, 3, 6$. So the amount of rain that has fallen is

$$\begin{aligned} R(1.5) &= \int_1^{1.5} \frac{1}{t^2}\, dt = -\frac{1}{t}\Big|_1^{1.5} = -1/1.5 - (-1) = .333 \\ R(3) &= \int_1^{3} \frac{1}{t^2}\, dt = -\frac{1}{t}\Big|_1^{3} = -1/3 - (-1) = .667 \\ R(6) &= \int_1^{6} \frac{1}{t^2}\, dt = -\frac{1}{t}\Big|_1^{6} = -1/6 - (-1) = .833 \end{aligned}$$

3. Example 2. When you exercise hard, lactic acid builds up in your muscles with a concentration measured in units mg/dL. Suppose that your rate of lactic acid production starting at time $t = 0$ min is given by $R(t) = 0.01t^2$; since it's a 'rate' that means it's a time derivative, so the units are (mg/dL)/ min. How much lactic acid concentration has built up after 30 minutes? Let's call the built-up concentration $C(t)$.

$$C(30) = \int_0^{30} 0.01t^2\, dt = 0.01 \cdot \frac{t^3}{3}\Big|_0^{30} = 0.01 \cdot \frac{30^3}{3} - 0 = 90$$

The units are mg/dL. Note that this is the increase over 30 minutes; if you started with a concentration of 5 mg/dL then after 30 minutes, you would be up to 95 mg/dL.

4. Example 3. A water holding tank has one inlet at the top and one outlet at the bottom. Water flows in through the inlet at the rate $3t + 4t^2$ m^3/min, and flows out at the rate $0.1t^3 + 6t^2$ m^3/min. Over the 10 minute span from $t = 0$ to $t = 10$, how much water has entered or left the tank?

There are 2 main ideas here.

- We are measuring the 'net accumulated volume of water'; I'll call it $W(t)$. If $W(t)$ is positive, there is a net gain, so the tank is filling, while if $W(t)$ is negative, more water is flowing out than is flowing in, so there is a net loss of water in the tank.
- The net rate of water gain/loss is given by the inflow rate minus the outflow rate. In general terms, this is

$$\text{net rate of change of volume} = \text{inflow rate} - \text{outflow rate}$$

This is called a 'conservation law' and is the basis for calculations involving conservation of mass, conservation of energy, conservation of momentum, and so on. If the inflow and outflow rates are equal, then the net rate of change is 0, and the quantity remains constant.

Now, the 'net rate of change' is a derivative - in this case, it is $W'(t) = (3t+4t^2)-(0.1t^3+6t^2)$. Then the accumulated volume of water is given by

$$\begin{aligned}
W(t) &= \int_0^{10} W'(t)\,dt \\
&= \int_0^{10} (3t+4t^2)-(0.1t^3+6t^2)\,dt \\
&= \int_0^{10} 3t-2t^2-0.1t^3\,dt \\
&= \frac{3}{2}t^2-\frac{2}{3}t^3-\frac{0.1}{4}t^4\,\Big|_0^{10} \\
&= 150-667-250=-767
\end{aligned}$$

The interpretation is that over the 10 minute period, the tank lost 767 m^3 of water.