Section 5.2 Riemann Sums

1. When we use a finite number of rectangles (N), we get an approximation to the area under a curve, and we call this a 'numerical sums' calculation. The notation we use is

$$A = \int_a^b f(x)\,dx \approx \sum_{i=1}^{N} f(x_i)\Delta x$$

Here, the height of each rectangle is $f(x_i)$ (using right endpoints) and the width is Δx.

2. To get the exact area, we take the limit $N \to \infty$. The rectangles get narrower and the error associated with each rectangle shrinks to zero. This is called a Riemann Sum.

3. The formula for the area is

$$A = \int_a^b f(x)\,dx = \lim_{N\to\infty} \sum_{i=1}^{N} f(x_i)\Delta x$$

This formula can be used directly only for simple polynomial functions, so it has little practical value for us. However, the idea is very important in Calculus II and III. The idea is that to find the area of a region you break it into simple pieces, find the area of each piece and then add the pieces together. There is an example below that uses the formula directly, but it is only for purposes of illustration.

4. Notice how the notation works. The Δx converts to the differential dx (remember, the differential is a small nonzero number) and the Σ for the sum is converted to the integral sign $\int$.

5. The formula for x_i is important only if we are doing a hand calculation. The formula is

$$x_i = a + \Delta x i = a + \frac{b-a}{N} i$$

6. Example. Write the Riemann Sum for the area under $f(x) = x^2 + x$ on the interval $[2, 3]$.

Start with finding x_i. The width of the interval is $b - a = 1$ so the formula is

$$x_i = 2 + \frac{1}{N} i$$

Then it is clear that

$$f(x_i) = x_i^2 + x_i = \left(2 + \frac{1}{N} i\right)^2 + \left(2 + \frac{1}{N} i\right)$$

We can insert this into the formula above to get the final form:

$$\begin{aligned} A = \int_a^b f(x)\,dx &= \lim_{N\to\infty} \sum_{i=1}^{N} f(x_i)\Delta x \\ \int_2^3 x^2 + x\,dx &= \lim_{N\to\infty} \sum_{i=1}^{N} \left(\left(2 + \frac{1}{N} i\right)^2 + \left(2 + \frac{1}{N} i\right)\right)\left(\frac{1}{N}\right) \end{aligned}$$

7. Illustrative Example. Let's compute the value of A in the example above.

   First, let's simplify the expression for $f(x_i)$:

$$f(x_i) = \left(2 + \frac{i}{N}\right)^2 + \left(2 + \frac{i}{N}\right) = \left(4 + 4\frac{i}{N} + \frac{i^2}{N^2}\right) + \left(2 + \frac{i}{N}\right) = 6 + \frac{5}{N}i + \frac{i^2}{N^2}$$

   Now multiply by $\Delta x = \dfrac{1}{N}$ and insert the result into the summation:

$$\sum_{i=1}^{N} f(x_i)\Delta x = \sum_{i=1}^{N} \left(\frac{6}{N} + \frac{5}{N^2}i + \frac{i^2}{N^3}\right)$$

   Next, we use the properties of Sigma notation to separate this into 3 sums, noting that i is the index and N is a constant:

$$\sum_{i=1}^{N} f(x_i)\Delta x = \frac{6}{N}\sum_{i=1}^{N} 1 + \frac{5}{N^2}\sum_{i=1}^{N} i + \frac{1}{N^3}\sum_{i=1}^{N} i^2$$

   If we're going to make any progress on this, we need to be able to evaluate the sums. Those formulas do exist (but only for simple expressions):

$$\begin{aligned}
\sum_{i=1}^{N} 1 &= 1 + 1 + 1 + \ldots + 1 = N \\
\sum_{i=1}^{N} i &= 1 + 2 + 3 + \ldots + N = \frac{N(N+1)}{2} \\
\sum_{i=1}^{N} i^2 &= 1 + 4 + 9 + \ldots + N^2 = \frac{N(N+1)(2N+1)}{6}
\end{aligned}$$

   Plug these into the main expression to get

$$\begin{aligned}
\sum_{i=1}^{N} f(x_i)\Delta x &= \frac{6}{N}\sum_{i=1}^{N} 1 + \frac{5}{N^2}\sum_{i=1}^{N} i + \frac{1}{N^3}\sum_{i=1}^{N} i^2 \\
&= \frac{6}{N}(N) + \frac{5}{N^2}\left(\frac{N(N+1)}{2}\right) + \frac{1}{N^3}\left(\frac{N(N+1)(2N+1)}{6}\right) \\
&= 6 + \frac{5}{2} \cdot \frac{N(N+1)}{N^2} + \frac{1}{6} \cdot \frac{N(N+1)(2N+1)}{N^3}
\end{aligned}$$

   Whew. Now it's time to take the limit as $N \to \infty$. This looks familiar! If we write it with x instead of N, the familiar idea is that

$$\lim_{x\to\infty} \frac{2x^2 - 3x + 4}{6x^2 + 13x - 2} = \frac{2}{6}$$

   because the numerator and denominator balance (recall, we used to say $f(x) \approx \dfrac{2x^2}{6x^2}$). When we finally take the limit, we get the actual area:

$$A = \int_2^3 x^2 + x\,dx = 6 + \frac{5}{2}\cdot(1) + \frac{1}{6}\cdot(2) = 6 + \frac{5}{2} + \frac{1}{3} = \frac{53}{6}$$

   Clearly, we don't want to spend the rest of our lives doing this!

8. Interpreting Riemann Sums

Let's take a moment to examine the structure of a Riemann Sum. Write the following Riemann Sum as an integral:

$$A = \lim_{N\to\infty} \sum_{i=1}^{N} \frac{8}{N}\left(\left(-2+\frac{8}{N}i\right)^2 + 3\left(-2+\frac{8}{N}i\right)\right)$$

We know the expression inside the summation must be $f(x_i)\Delta x$ so let's look for those elements. Since Δx has the form $\frac{b-a}{N}$, that's easy to pick out: $\Delta x = \frac{8}{N}$. This means that $b-a=8$ so the width of the interval is 8. Once we know a, we can add 8 to get b.

Next, look for x_i, which has the form $a+\frac{b-a}{N}i$. Ah ha, we see $-2+\frac{8}{N}i$. This means that $a=-2$ so $b=6$. The structure in parenthesis is $f(x_i)$, and it is $x_i^2+3x_i$ so $f(x)=x^2+3x$. Finally, the Riemann Sum is the area under x^2+3x on the interval $[-2,6]$ which is

$$A = \int_{-2}^{6} x^2 + 3x\,dx$$