Section 4.9: Projectile motion.

This is the reason Calculus was invented! You've heard the story about the apple falling on Isaac's head. The apple is a projectile (an object subjected only to external forces, like gravity). The starting point is Newton's Second Law: $F = ma$. F is the set of forces acting on an object (right now, it's only gravity), m is the mass of the object, and a is the acceleration of the object.

That's right, forces cause objects to move. Without forces, there is no motion.

1. Standard Notation.

   The position of an object is denoted $s(t)$.

   The velocity of an object is denoted $v(t)$, which is equivalent to $s'(t)$.

   The acceleration of an object is denoted $a(t)$. It is the rate of change of velocity, so

   $$a(t) = v'(t) = s''(t)$$

2. The force of gravity

   The force of gravity is the mass m times the gravitational constant g, which equals 9.812 m/s^2 or 32.12 ft/s^2. We put a minus sign in front to signify that gravity pulls objects down, in the negative direction. This means that

   $$\begin{aligned} F &= ma \\ -mg &= ma \\ -g &= a \end{aligned}$$

   Ah ha, objects falling under the influence of gravity are subjected to a constant force, $-g$. We will usually use $g = 9.8$ (or sometimes $g = 10$ to make the number simple) in MKS units, or $g = 32$ in imperial units, because we are more concerned with the process rather than huge numbers of significant digits.

3. Projectile Motion problems

   The goal of these problems is to find the expression for the position $s(t)$ of an object. We start with $a = -g$, then take an antiderivative to get $v(t)$. The constant c is determined by the initial velocity; we'll write it as $v(0) = v_0$. Then we take another antiderivative to get $s(t)$. The constant d is determined by the initial position, written as $s(0) = s_0$. Here's how it works in general:

   $$\begin{aligned} s''(t) = a(t) &= -g \\ s'(t) = v(t) &= -gt + c = -gt + v_0 \\ s(t) &= -\frac{1}{2}gt^2 + v_0t + d = -\frac{1}{2}gt^2 + v_0t + s_0 \end{aligned}$$

   This formula for $s(t)$ is often called the equation of motion of the object.

4. In Calc I, we look only at one-dimensional motion - the object eithers moves straight up and down, or moves in a straight line. In Calc III, we are able to move into the three-dimensional world. In Differential Equations, we can add additional forces (air resistance, spring forces [Hooke's Law], electric and/or magnetic forces, etc).

5. Example of up/down projectile motion.

   A rocket is fired off a 10 ft platform with initial velocity 50 ft/s. Find the equation of motion and the maximum height achieved by the rocket.

   Since units are given in feet, we are using imperial units. The initial position of the rocket is $s(0) = 10$ (we are assuming the ground is at position $s = 0$), and the initial velocity is $v(0) = 50$. We begin with

$$\begin{aligned} a(t) &= -32 \\ v(t) &= -32t + c \\ & \quad 50 = v(0) = -32 \cdot 0 + c, \qquad c = 50 \\ v(t) &= -32t + 50 \\ s(t) &= -32\left(\frac{t^2}{2}\right) + 5t + d = -16t^2 + 50t + d \\ & \quad 10 = s(0) = -16 \cdot 0 + 50 \cdot 0 + d, \qquad d = 10 \end{aligned}$$

   The equation of motion is $s(t) = -16t^2 + 50t + 10$.

   To find the maximum height, note that this is the maximum value of the position function $s(t)$. This is located at a critical point of $s(t)$. The process is to set the derivative of $s(t)$ equal to zero, find the time at which the critical point occurs, and plug that t value into the function s to find its maximum value. But we already have the derivative of $s(t)$, it's $v(t)$. So

$$\begin{aligned} 0 &= v(t) = -32t + 50 \\ t &= 50/32 = 1.5625 \end{aligned}$$

   The maximum height is $s(1.5626) = -16 \cdot 1.5625^2 + 50 \cdot 1.5625 + 10 = 49.0625$.

6. Example of straight line motion

   A car travelling at 50 mph hits the brakes and decelerates at the rate of 40 ft/s^2. What is the stopping distance?

   Here, the force is not gravity, but the deceleration of the car, so $a = -40$. We should measure the stopping distance using feet rather than miles, so let's convert the speed from miles per hour to feet per second:

$$50\ \frac{\text{miles}}{\text{hour}} = 50\ \frac{5280\text{ feet}}{3600\text{ second}} = 73.3\ \frac{\text{feet}}{\text{second}}$$

   This is the initial velocity: $v(0) = 73.3$. For convenience, let's set the initial position of the car to zero ($s(0) = 0$); that is the instant that the brakes are first applied. The calculation goes like this:

$$\begin{aligned} a(t) &= -40 \\ v(t) &= -40t + c \\ &\quad 73.3 = v(0) = -40 \cdot 0 + c = c, \qquad c = 73.3 \\ s(t) &= -40\left(\frac{t^2}{2}\right) + 73.3t + d = -20t^2 + 73.3t + d \\ &\quad 0 = s(0) = -20 \cdot 0^2 + 73.3 \cdot 0 + d, \qquad d = 0 \end{aligned}$$

   The equation of motion of the car is $s(t) = -20t^2 + 73.3t$.

   Now we have 3 formulas - for acceleration, velocity and position. To find the stopping distance, we need to identify which formulas to use. The stopping distance refers to the position of the car, so we need to compute $s(t)$. But at what time does this happen? That t value is what we will plug into $s(t)$.

   To find the time at which the car stops, we set the velocity to zero:

$$\begin{aligned} 0 &= v(t) = -40t + 73.3 \\ t &= 73.3/40 = 1.83 \end{aligned}$$

   It takes the car 1.83 seconds to come to a stop. The position at that time is $s(1.83) = -20 \cdot 1.83^2 + 73.3 \cdot 1.83 = 67.2$ ft.