Section 4.9: Antiderivatives

1. We started the course with limits, and have studied derivatives for a good chunk of the semester. Now it is time for us to study the third main topic of calculus - integrals. Derivatives tell us the instantaneous rate of change of a quantity. Integrals tell us how a quickly a quantity accumulates. Derivatives and integrals are related to each other through the notion of antiderivatives. Pronounce 'anti' as ant eye rather than auntie.

2. The way to think about antiderivatives is this: if you have $f'(x)$, what is the function $f(x)$ that it came from?

3. What function $f(x)$ did $f'(x) = 2x$ come from? It should be apparent that the answer is $f(x) = x^2$. But wait, what about $f(x) = x^2 + 1$ or $f(x) = x^2 - 37$? All of those are valid choices, because the derivative of a constant is zero.

4. The way we write this is $f(x) = x^2+c$. This is called 'the most general antiderivative', and the c is called the 'constant of integration' in Chapter 5. For now, we don't have a specific value for c, we just know that there could be such a constant in $f(x)$. In Differential Equations, after Calculus III, these constants become vital, so you need to get into the habit now of writing them for every single problem.

5. What function $f(x)$ did $f'(x) = x^2$ come from? It's easy to see that there is an x^3 lurking about, but if we try $f(x) = x^3$, the derivative is $f(x) = 3x^2$. We have to compensate for the 3, so we adjust our choice for $f(x)$ to $f(x) = \dfrac{1}{3}x^3 + c$. Then the derivative is $f'(x) = \dfrac{1}{3}(3x^2) + 0 = x^2$.

6. Let's start to be systematic here. Examine each example below to make sure each one makes good sense. Do this by differentiating each $f(x)$ to see that the derivatives does match $f'(x)$. The process of going from $f(x)$ to $f'(x)$ (right to left in the table) is called differentiation, while the process of going from $f'(x)$ to $f(x)$ (left to right in the table) is called antidifferentiation.

(a) $f'(x) = x$	$f(x) = \frac{1}{2}x^2 + c$
(b) $f'(x) = 1$	$f(x) = x + c$
(c) $f'(x) = 0$	$f(x) = 0 + c = c$
(d) $f'(x) = x^n$	$f(x) = \frac{1}{n+1}x^{n+1} + c$
(e) $f'(x) = \cos x$	$f(x) = \sin x + c$
(f) $f'(x) = \sec^2 x$	$f(x) = \tan x + c$
(g) $f'(x) = \sec x \tan x$	$f(x) = \sec x + c$
(h) $f'(x) = e^x$	$f(x) = e^x + c$
(i) $f'(x) = \frac{1}{x}$	$f(x) = \ln x + c$
(j) $f'(x) = \frac{1}{1+x^2}$	$f(x) = \tan^{-1} x + c$
(k) $f'(x) = \cosh x$	$f'(x) = \sinh x + c$

7. Item 6(d) is called the power rule, just as we used in Section 3.1 when we learned to take derivatives of power functions. Recall that the derivative of x^n is nx^{n-1}. In words, multiply by the power and subtract one from the power. For the antiderivative, we do the inverse operations: add one to the power, then divide by that number. Items (a), (b) and (c) are special cases of the general rule.

8. This means that the antiderivative of $f'(x) = x^{6/7}$ is $f(x) = \dfrac{1}{13/7}x^{13/7} + c = \dfrac{7}{13}x^{13/7} + c$. You'll need to be able to do this quickly, so you need to practice. Work the suggested problems in the homework on WebAssign, and all of the practice problems I will give you.

9. Note that Item 6(b) can be generalized, because when we multiply by a constant, it just carries along:

$$f'(x) = a \quad f(x) = ax + c$$

10. What is the antiderivative of $f'(x) = x(x^2+1)^{-1/2}$? Not so obvious, eh? But if I tell you that $f(x) = (x^2+1)^{1/2} + c$, you can verify it by differentiating $f(x)$. You'll learn how to do this yourself in Section 5.5.

11. The first half of Calculus II is how to find antiderivatives for more complicated functions $f'(x)$, so we need to be sure that you learn this foundational material well this semester.

Now we can do some application problems to give you a feel for how we use these things.

Ex 1. Suppose that $f'(x) = x$ and we know that $f(1) = 2$. Find $f(x)$.

The statement $f(1) = 2$ is called an initial condition. It is used to find the value of c used in this problem. First, use the power rule (6d) to get the general antiderivative: $f(x) = \frac{1}{2}x^2 + c$. Then, 'plug in' the initial condition. It says that the function equals 2 when x equals one, so we substitute those values in:

$$\begin{aligned} f(x) &= \frac{1}{2}x^2 + c \\ 2 &= \frac{1}{2}1^2 + c \\ \frac{3}{2} &= c \end{aligned}$$

so the final answer is $f(x) = \frac{1}{2}x^2 + \frac{3}{2}$.

Ex 2. If we start with a function $f(x)$, we can take its derivatives easily. For example,

$$\begin{aligned} f(x) &= 4 + 2x - 5x^2 \\ f'(x) &= 2 - 10x \\ f''(x) &= -10 \end{aligned}$$

Now, let's go backwards. Let's start with a function $h''(x)$, then find its antiderivative $h'(x)$, then find the next antiderivative $h(x)$:

$$\begin{aligned} h''(x) &= -6 \\ h'(x) &= -6x + c \\ h(x) &= -6\left(\frac{x^2}{2}\right) + cx + d = -3x^2 + cx + d \end{aligned}$$

Notice that the cx came from rule (9). The reason that d is there is that each time we take an antiderivative, we need to add a constant, and the symbol c was already being used.

Ex 3. Suppose $f''(x) = \frac{1}{6}x^{1/3}$ with initial conditions $f'(0) = 2$ and $f(0) = 3$. Find $f(x)$.

First, take the antiderivative of $f''(x)$, which gives $f'(x)$.

$$\begin{aligned} f'(x) &= \frac{1}{6}\left(\frac{x^{4/3}}{4/3}\right) + c \\ &= \frac{3}{4} \cdot \frac{1}{6}x^{4/3} + c = \frac{1}{8}x^{4/3} + c \end{aligned}$$

Now we can find the value for c because we know $f'(0) = 2$. Plug in $x = 0$ and $f' = 2$ to get

$$2 = f'(0) = \frac{1}{8}0^{4/3} + c = c$$

So $c = 2$ and

$$f'(x) = \frac{1}{8}x^{4/3} + 2$$

Now, take the antiderivative again to get $f(x)$:

$$f(x) = \frac{1}{8}\left(\frac{x^{7/3}}{7/3}\right) + 2x + d = \frac{3}{56}x^{7/3} + 2x + d$$

Apply the condition $f(0) = 3$ by plugging in $x = 0$ and $f = 3$ to get

$$3 = f(x) = \frac{3}{56}0^{7/3} + 0 + d = d$$

So $d = 3$ and the final form of $f(x)$ is

$$f(x) = \frac{1}{8}\left(\frac{x^{7/3}}{7/3}\right) + 2x + 3$$