Section 4.7: Optimization

1. This is one of the main applications of calculus, so it is very important. Optimization means either maximizing or minimizing a quantity. Businesses want to maximize profit or minimize costs, engineers want to build a contraption using the least amount of materials (that's minimizing volume or surface area), students want to maximize their course grade.

2. Real-world optimization problems are huge. For example, when a national trucking company wants to optimize its driver routes and decide which trucks carry what to where, there can be half a million variables to consider.

3. But let's not go there immediately. Instead, let's start with problems involving 2-4 variables.

4. In any application, there has to be a variable that is supposed to be maximized or minimized. For example, consider the volume of a box. The biggest volume you can imagine is infinite, but that's not very helpful. To make the problem meaningful, there have to be *constraints*, which are conditions placed on the problem that would limit the variable we are trying to optimize. For the volume of the box, a realistic constraint would be the amount of cardboard we have available to build the box. Note that we need to be precise - 'amount' means surface area.

5. So, the elements of an optimization problem are

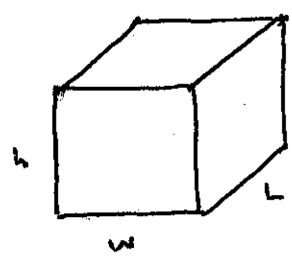
   - A quantity to be optimized. This is written in terms of a function, which might start off with more than one independent variable. For example, the volume of a box is $V = hwL$, height times width times length.
   - One or more constraints need to be present in order to rewrite the variable in terms of a single independent variable. For the box, if the base is supposed to be square, then $w = L$, so that $V = hw^2$. One more constraint that relates h and w would be needed to complete the setup.

6. Once we have the quantity written in terms of a single variable, it becomes a calculus problem. A max/min point is located at a crticial point, so we find the critical points of our function by setting the derivative to zero. Be sure to answer the original question with a brief statement.

Example 1. Let's build a box that has a square base and no top to store our supply of toilet paper. The box must have volume 10 m^3. Find the dimensions of the box that uses the minimum amount of material.

Here is the thought process:

1. The underlying quantities are the volume, so $V = hwL$, and the surface area of the box. To get a formula for the surface area, note that there is the bottom, with area wL, and 4 sides. Two of those sides have area hL and two have area wL, as shown in the amazing sketch below. This means the total surface area is $A = wL + 2hL + 2wh$.



Front, back: wh + wh

Left, right: hL + hL

bottom: wL

2. What is it that we are supposed to optimize? The phrase 'minimum amount of material' means that we are supposed to minimize the total surface area. Therefore, we need to rewrite A so that it depends on only one variable by using constraints. Since A depends on 3 variables, we need to 2 constraints to eliminate 2 of those variables. The decision of which variables to eliminate will depend on the algebraic structure of the formula for A; sometimes, one choice leads to a slightly easier expression, but it really doesn't matter - we'll end up in the right place no matter which ones we eliminate.

3. The first constraint we can identify is that the base is square, so $w = L$. We can eliminate either of those from the problem; I choose to eliminate w, so I can write

$$\begin{aligned} V &= L^2h \\ A &= L^2 + 4Lh \end{aligned}$$

4. Now we need another constraint to eliminate either L or h. The statement is that the volume must be 10 m^3. Ok, so the constraint is

$$10 = L^2h$$

5. Should we eliminate L or h? Consider these two options:

$$h = \frac{10}{L^2} \qquad L = \sqrt{\frac{10}{h}}$$

It seems more natural to avoid the square root, so let's choose to eliminate h by inserting the first relation into the expression for the surface area:

$$A = L^2 + 4Lh = L^2 + 4L\left(\frac{10}{L^2}\right) = L^2 + \frac{40}{L}$$

6. Now we have the area written in terms of a single variable, so we can apply calculus. Looking at the graph of A, we can see that there is clearly a single minimum value, and we are looking for the value of L that gives it.

You can interpret the graph this way. If L is too small, then the square base is tiny so the box needs to be really tall to have volume 10 m^3; this takes too much surface area on the sides. If L is too big, then the base is big. The height will be small but the large base will take too much surface area. The minimum total area occurs where there is a balance between the area of the base and the area of the sides.

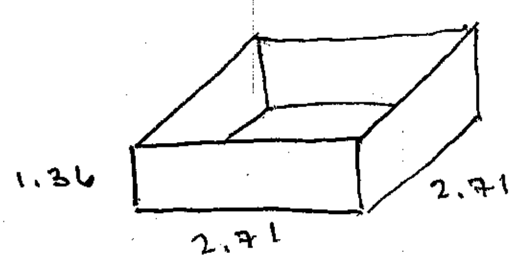
7. Let's do the calculus! Take the derivative and set it to zero.

$$\begin{aligned} A &= L^2 + \frac{40}{L} \\ \frac{dA}{dL} &= 2L - \frac{40}{L^2} = 0 \\ 2L &= \frac{40}{L^2} \\ L^3 &= 20 \\ L &= 2.71 \end{aligned}$$

The units are m.

8. To finish the problem, we need to 'find the dimensions'. That means we need to report w and h as well. Go back to the earlier relations to do this. From item (3), $w = L$, so $w = 2.71$ m too. From item (5), $h = \frac{10}{L^2} = \frac{10}{2.71^2} = 1.36$ m.

Our final report is this: The dimensions of the box should be $2.71 \times 2.71 \times 1.36$ ~~cm~~ m.



Example 2. Find the minimum distance between the point $(1, 4)$ and the line $3x + 2y = 7$ in the xy plane.

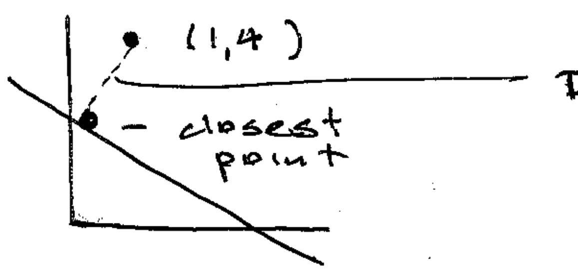
1. As an aside, this purely geometric problem can be extended to have real applications. For example, you are standing at the point, and a drone is flying along the path of the line. How close does the drone get to you? In other words, will the drone detect you, so that the feds know you are not in your house when you are supposed to be?

2. Ok, back to the problem. We are supposed to find the minimum distance between the point and the line. The line is composed of many points that can be represented in general as (x, y). So the distance between $(1, 4)$ and a point on the line (x, y) (let's call it D) comes from the distance formula:

$$D^2 = (x-1)^2 + (y-4)^2$$

We could take the square root, but we are going to differentiate, and the square root on the right would complicate things. Instead we note that minimizing the square of the distance is the same as minimizing the distance to avoid this issue. Let's relabel this as

$$F = (x-1)^2 + (y-4)^2$$



3. But now we need a constraint to eliminate either x or y. The constraint is that we can't use any old point (x, y), it needs to be on the line. The equation of the line is the constraint. This means that

$$y = -\frac{3}{2}x + \frac{7}{2}$$

Now we can write any point on the line as having coordinates $\left(x, -\frac{3}{2}x + \frac{7}{2}\right)$ and the distance formula can be written as

$$F = (x-1)^2 + (y-4)^2 = (x-1)^2 + \left(-\frac{3}{2}x - \frac{1}{2}\right)^2$$

Don't multiply this out! Instead, we will use the chain rule.

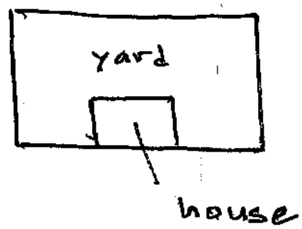
4. To find the critical point, set the derivative to zero:

$$
\begin{aligned}
0 = F'(x) &= 2(x-1) + 2\left(-\frac{3}{2}x - \frac{1}{2}\right)\left(-\frac{3}{2}\right) \\
0 &= 2(x-1) - 3\left(-\frac{3}{2}x - \frac{1}{2}\right) \\
0 &= 2x - 2 + \frac{9}{2}x + \frac{3}{2} \\
0 &= \frac{13}{2}x - \frac{1}{2} \\
\frac{13}{2}x &= \frac{1}{2} \\
x &= \frac{1}{13} \\
y &= -\frac{3}{2}x + \frac{7}{2} = \frac{16}{13}
\end{aligned}
$$

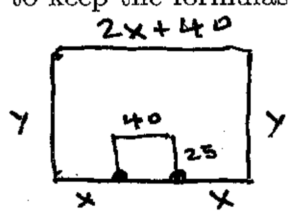
The point on the line nearest to $(1,4)$ is $\left(\frac{1}{13}, \frac{16}{13}\right)$ and the minimum distance is given by

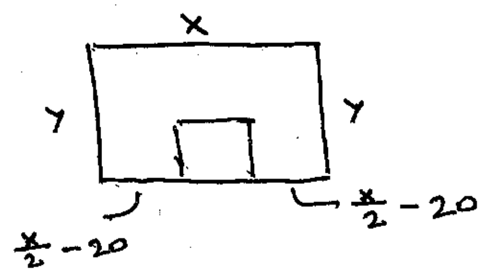
$$
\begin{aligned}
F = D^2 &= (x-1)^2 + (y-4)^2 \\
&= \left(\frac{1}{13} - 1\right)^2 + \left(\frac{16}{13} - 4\right)^2 \\
&= \frac{144}{169} + \frac{64}{169} = \frac{208}{169} \\
D &= \sqrt{\frac{208}{169}} = 1.1094
\end{aligned}
$$

Example 3. Let's build a rectangular fence to enclose the back yard as shown below. The house is 40 ft wide and 25 ft deep. We have 500 ft of fence. What dimensions should be used to maximize the enclosed area?



5. First, we need to pick variables to represent the dimensions of the fence to build the formula for the area. There are 2 reasonable choices, shown below. We'll pick the option on the left to keep the formulas as simple as possible.





6. The enclosed area is the area of the outer rectangle minus the area of the house:

$$A = (2x + 40)y - 40 * 25$$

7. We need a constraint to eliminate either x or y. Ah ha, the perimeter of the fence is 500 ft. The fence has 5 components - the left side, the top, the right side, the bottom left and the bottom right. Together, they form the constraint equation:

$$500 = 2y + (2x + 40) + 2x$$

so that $y = 230 - 2x$. Now we can write the area as

$$A = (2x + 40)(230 - 2x) - 1000$$

8. Now set the derivative to zero. Use the product rule rather than multiplying out the quadratic.

$$\begin{aligned} A' = 0 &= [2](230 - 2x) + (2x + 40)[-2] \\ 0 &= 460 - 4x - 4x - 80 \\ 0 &= 380 - 8x \\ x &= 47.5 \\ y &= 230 - 2x = 135 \end{aligned}$$

The final dimensions look like this. Hey, the overall figure is a square!