Section 4.5 part A: summary of curve sketching

The goal here is to use the features of $f(x)$, $f'(x)$ and $f''(x)$ to draw a qualitative sketch of $f(x)$ showing all of its pertinent features (critical points, max/min values, inflection points, vertical asymptotes and 'end behavior' - what the graph looks like on the far left and far right). Some examples of behavior are $\lim_{x\to\infty} f(x) = \infty$ or $\lim_{x\to-\infty} f(x) = 3$.

There are 2 reasons to do this: (1) a computer generated image might not show all of those features if they are subtle, (2) you are building cognitive complexity by combining bits of information from different sources and synthesizing them into a package. The second reason is more important for us.

There are 3 types of problems in this section, listed below. This reminds me of one of my best jokes: There are three kinds of people in the world - those who can count and those who can't. I am missing the opportunity to tell you that joke to your faces.

A. Given a function $f(x)$, find its derivatives, identify critical points and possible inflection points, build the sign chart and sketch the graph. These problems can easily be quite lengthy, so we won't do a lot of them. But do be mindful of each step so that you see the role it plays in the graph. Q15 is an example.

B. Given a list of information about $f(x)$, $f'(x)$ and $f''(x)$, build the sign chart and sketch the graph. These problems avoid the algebra of simplifying the derivatives so they are more manageable. Q13 problem 2 is an example.

C. Use roots and end behavior to draw a quick sketch of $f(x)$ without using derivatives. This is quick and easy but won't tell us the location of max/min values or inflection points.

Finding End Behavior

1. Here, we are looking at the behavior of a function as $x \to \infty$ and $x \to -\infty$. This is why we did limits in the previous section.

2. We will restrict our analysis to functions with powers, so we won't see the complicated structures we worked on using l'Hospital's Rule.

3. The main point is that the highest power in the numerator and the highest power in the denominator will determine what happens.

4. For example, in the expression $5x^3 - 2x + 7$, the $5x^3$ is the 'dominant term', and dictates how the expression behaves. To see this, consider $x = 100$ (not exactly infinity but close enough if we think about a viewing window of $x \in [-10, 10]$ for a graph). The term $5x^3 = 5,000,000$, the term $2x = 200$ and the term 7 is just 7. Clearly the $5x^3$ is the biggest term - the others are inconsequential. This means we can use the approximation $5x^3 - 2x + 7 \approx 5x^3$, which is valid when x has a large magnitude.

5. This analysis tells us nothing about the shape of the graph in its 'central area'. To find out what happens there, we need the information that comes from the derivatives.

Example 1. Consider $f(x) = \dfrac{5x^3 - 2x + 7}{2x + 3}$.

First, since there are no square roots we can do both $+\infty$ and $-\infty$ together. In the numerator, the dominant term is $5x^3$ and in the denominator the dominant term is $2x$. This means we can make a quick and easy approximation:

$$\frac{5x^3 - 2x + 7}{2x + 3} \approx \frac{5x^3}{2x} = \frac{5}{2}x^2$$

So the function behaves like x^2 - it goes to $+\infty$ on the left and the right. In the central region, we would need to look at its derivatives to see what is happening.

Example 2. Consider $f(x) = \dfrac{5x^3 - 2x + 7}{3 - 2x^2}$.

That change in the denominator has a big impact. Now, we have

$$\frac{5x^3 - 2x + 7}{3 - 2x^2} \approx \frac{5x^3}{-2x^2} = \frac{-5}{2}x$$

This function looks like a line with negative slope on the far left and far right, so it goes to $+\infty$ on the left and $-\infty$ on the right.

Example 3. Consider $f(x) = \dfrac{\sqrt{16x^6 - x}}{2x^3 + 1}$.

We did some of these in the context of limit problems back in Chapter 2. With a square root, there is the possibility that the analysis will be different for $x \to \infty$ and $x \to -\infty$, so we need to analyze them separately.

Recall that a square root is always positive, so we use absolute values when we simplify:

$$\begin{aligned}
\sqrt{x^2} &= |x| = \text{either } +x \text{ or } -x \\
\sqrt{x^4} &= |x^2|, \text{ which is always } x^2 \\
\sqrt{x^6} &= |x^3| = \text{either } +x^3 \text{ or } -x^3
\end{aligned}$$

Behavior at the far right ($x \to \infty$). We approximate the numerator and denominator using dominant powers:

$$f(x) = \frac{\sqrt{16x^6 - x}}{2x^3 + 1} \approx \frac{\sqrt{16x^6}}{2x^3} = \frac{4|x^3|}{2x^3} = \frac{4x^3}{2x^3} = 2$$

Two important notes:

- Since x is positive here, so is x^3, so $|x^3| = x^3$.
- We are not saying that $f(x) = 2$, we are saying that as $x \to \infty$, $f(x)$ gets closer and closer to 2. On the right end of the graph, there is a horizontal asymptote $y = 2$.

Behavior at the far left ($x \to -\infty$). Now x is negative, so x^3 is negative, so $|x^3| = -x^3$. Now we have

$$f(x) = \frac{\sqrt{16x^6 - x}}{2x^3 + 1} \approx \frac{\sqrt{16x^6}}{2x^3} = \frac{4|x^3|}{2x^3} = \frac{-4x^3}{2x^3} = -2$$

At the left end of the graph, there is a horizontal asymptote $y = -2$.