Section 4.4 E

Limits of the form ∞^0, 1^∞, 0^0, 0^∞

Start with $L = \lim_{x \to a} f(x)^{g(x)}$

$$= \lim_{x \to a} e^{\ln\left(f(x)^{g(x)}\right)}$$

$$= \lim_{x \to a} e^{g(x) \ln f(x)}$$

$\ln(A^B)$
$= B \ln A$

or

$$\lim_{x \to a} \exp\left(g(x) \ln f(x) \right)$$

$$= \exp\left(\lim_{x \to a} g(x) \ln f(x) \right)$$

Let's write this as $L = e^K$

where $K = \lim_{x \to a} g(x) \ln f(x)$

This avoids writing 'exp' too many times, and let's us focus on that product $g(x)$ times $\ln(f(x))$

Ex 16 $L = \lim_{x\to\infty} x^{1/x}$ form : ∞^0

The base x wants the limit to be ∞

The exponent $1/x$ wants the limit to be 1

$$L = \lim_{x\to\infty} e^{\ln(x^{1/x})} = \lim_{x\to\infty} e^{\frac{1}{x}\cdot\ln x}$$

$$= \exp\left(\lim_{x\to\infty} \frac{1}{x}\cdot\ln x\right)$$

write : $L = e^{K}$, $K = \lim_{x\to\infty} \frac{1}{x}\cdot\ln x$

and focus on K

$K = \lim_{x\to\infty} \frac{1}{x}\cdot\ln x$ $\frac{1}{\infty}\cdot\infty = 0\cdot\infty$ so

write as a fraction

$K = \lim_{x\to\infty} \frac{\ln x}{x}$ $\frac{\infty}{\infty}$

$$\stackrel{L'H}{=} \lim_{x\to\infty} \frac{1/x}{1} = \frac{1/\infty}{1} = \frac{0}{1} = 0$$

so

$$L = e^{K} = e^{0} = 1$$

$\left(x^{1/x} \to 1 \right)$

note : the exponent beat the base this time

Ex 17 $L = \lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x$ form $(1+0)^\infty = 1^\infty$

The base wants the limit to be 1

The exponent wants the limit to be ∞

$$L = \lim_{x\to\infty} e^{\ln\left(1+\frac{1}{x}\right)^x} = \lim_{x\to\infty} e^{x\ln\left(1+\frac{1}{x}\right)}$$

$$= e^K \quad \text{where}$$

$K = \lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right)$ form $\infty\cdot\ln 1 = \infty\cdot 0$ so rewrite as a fraction

$$K = \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{1/x} \qquad \frac{\ln(1)}{1/\infty} = \frac{0}{0} \quad \text{so L'H}$$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{1+1/x}\cdot\left(-1/x^2\right)}{-1/x^2} \qquad \leftarrow \text{chain rule}$$

$$(x^{-1})' = -x^{-2} = -1/x^2$$

cancel $-1/x^2$

$$K = \lim_{x\to\infty} \frac{1}{1+1/x} = \frac{1}{1+1/\infty} = \frac{1}{1+0} = 1$$

so $L = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e^K = e^1 = e$

This is one way to define e

Ex 18 $L = \lim_{x\to 0^+} (\tan 2x)^x \qquad (0^0)$

$(x>0)$

$= \lim_{x\to 0^+} e^{\ln(\tan 2x)^x} = \lim_{x\to 0^+} e^{x\ln(\tan 2x)}$

$= e^k$ where

$k = \lim_{x\to 0^+} x\ln(\tan 2x)$ form $0\cdot\ln(0)$

$0\cdot(-\infty)$ so L'H

$k = \lim_{x\to 0^+} \dfrac{\ln(\tan 2x)}{1/x} \qquad \left(\dfrac{\ln 0}{1/0} = \dfrac{-\infty}{\infty}\right)$

$\overset{L'H}{=} \lim_{x\to 0^+} \dfrac{\frac{1}{\tan 2x}\cdot\sec^2(2x)\cdot 2}{-1/x^2}$ ← chain rule

. note: $\dfrac{\sec^2(2x)}{\tan(2x)} = \dfrac{\frac{1}{\cos^2(2x)}}{\frac{\sin(2x)}{\cos(2x)}} = \dfrac{1}{\cos^2(2x)}\cdot\dfrac{\cos 2x}{\sin 2x}$

$= \dfrac{1}{\sin 2x\cdot\cos 2x}$

. recall the trig identity $\sin(2A) = 2\sin A\cos A$

so $\sin A\cos A = \frac{1}{2}\sin(2A)$

$\sin 2x\cdot\cos 2x = \frac{1}{2}\sin(4x)$

. dump this into the numerator

$k = \lim_{x\to 0^+} \dfrac{\left(\frac{1}{\frac{1}{2}\sin(4x)}\right)\cdot 2}{-1/x^2} = \lim_{x\to 0^+} \dfrac{\frac{4}{\sin(4x)}}{-1/x^2}$

$= \lim_{x\to 0^+} \dfrac{-4x^2}{\sin(4x)}$ form: $\dfrac{0}{0}$

$$k = \lim_{x \to 0^+} \frac{-4x^2}{\sin 4x} \qquad \frac{0}{0}$$

$$\overset{L'H}{=} \lim_{x \to 0^+} \frac{-8x}{\cancel{4\cos(4x)}} = \frac{-0}{4 \cdot 1} = 0$$

So $L = \lim_{x \to 0^+} (\tan 2x)^x = e^k = e^0 = 1$

Try these on your own

Ex 19 $\quad L = \lim_{x \to 1^+} \left[\ln(x^7 - 1) - \ln(x^5 - 1) \right]$

Ex 20 $\quad L = \lim_{x \to \infty} x^{e^{-x}}$

Ex 21 $\quad L = \lim_{x \to 0} \frac{\sin^{-1} x}{x}$

Ex 22 $\quad L = \lim_{x \to \infty} (x - \ln x)$