Section 4.4 B

Limits of the form $\infty - \infty$

The trick is to look at individual components in a problem, then to see how the components fit together

Ex 1 $L = \lim_{x \to \infty} x^2 - x^3$

components : $x^2 \to \infty$ and $x^3 \to \infty$

fit together : $\infty - \infty$; which one dominates?

technique : convert to a product

$L = \lim_{x \to \infty} x^2(1-x)$

components : $x^2 \to \infty$, $1-x \to -\infty$

fit together : $\infty \cdot (-\infty)$

technique : recognize this now determinate : $\infty \cdot (-\infty) = -\infty$

Writing it out :

$L = \lim_{x \to \infty} x^2 - x^3$ $\quad \infty - \infty$

$= \lim_{x \to \infty} x^2(1-x)$

$= \infty \cdot (-\infty)$

$= -\infty$

Ex 2 $L = \lim_{x \to 1^+} \left(\frac{1}{\ln x} - \frac{1}{x-1} \right)$

components : $\frac{1}{\ln x} \to \frac{1}{0} = \infty$

$\frac{1}{x-1} \to \frac{1}{1-1} = \frac{1}{0} = \infty$

(we use 1^+ so that $x > 1$ so $\frac{1}{x-1} \to +\infty$; if we said 1^- then $x < 1$ and $\frac{1}{x-1} \to -\infty$; the limit would be $\infty - (-\infty) = +\infty$)

fit together : $\infty - \infty$, which one dominates?

technique : get common denominator to form a fraction

$$L = \lim_{x \to 1^+} \left(\frac{1}{\ln x} \cdot \frac{x-1}{x-1} - \frac{1}{x-1} \cdot \frac{\ln x}{\ln x} \right)$$

$$= \lim_{x \to 1^+} \frac{x-1-\ln x}{(x-1)\ln x}$$

components : $x-1-\ln x \to 1-1-\ln 1 = 1-1-0 = 0$

$(x-1)\ln x \to (1-1)\ln 1 = 0 \cdot 0 = 0$

fit together : $\frac{0}{0}$ hey, we can apply L'H !

to notate, we write

$$L = \lim_{x \to 1^+} \frac{x-1-\ln x}{(x-1)\ln x}$$

$$\overset{L'H}{=} \lim_{x \to 1^+} \frac{1-0-1/x}{[1]\ln x + (x-1)\frac{1}{x}} \leftarrow \frac{f'(x)}{g'(x)} \leftarrow \text{product rule}$$

note that we are using L'H

- components : $1 - \frac{1}{x} \to 1 - \frac{1}{1} = 0$

$\ln x + \frac{x-1}{x} \to \ln 1 + \frac{1-1}{1} = 0 + 0 = 0$

- hey, this is still $\frac{0}{0}$ so apply L'H again

$L = \lim_{x \to 1^+} \dfrac{1 - \frac{1}{x}}{\ln x + \frac{x-1}{x}}$ ← hey, multiply by $\frac{x}{x}$
then apply L'H

$= \lim_{x \to 1^+} \dfrac{x-1}{x\ln x + x - 1}$

$\overset{L'H}{=} \lim_{x \to 1^+} \dfrac{1}{[1]\ln x + x[\frac{1}{x}] + 1}$ $\dfrac{f'}{g'}$ (product rule)

top $\to 1$
bottom $\to \ln 1 + 1 + 1 = 2$
no longer indeterminate so

$L = \lim_{x \to 1^+} \left(\dfrac{1}{\ln x} - \dfrac{1}{x-1} \right) = \dfrac{1}{2}$

Ex 3 $L = \lim_{x \to \infty} \dfrac{\sinh x}{\cosh x}$

top $= \frac{1}{2}(e^x - e^{-x}) \to \frac{1}{2}(\infty - 0) = \infty$
bottom $= \frac{1}{2}(e^x + e^{-x}) \to \frac{1}{2}(\infty + 0) = \infty$
fit: $\frac{\infty}{\infty}$ so apply L'H

$L \overset{L'H}{=} \lim_{x \to \infty} \dfrac{\cosh x}{\sinh x}$ $(\sinh x)' = \cosh x$
$(\cosh x)' = \sinh x$

hey, this still gives $\frac{\infty}{\infty}$.
L'H was the wrong approach

Recall, we did problems like this in Chapter 2

If L'H doesn't work, then divide top and bottom by the thing causing the ∞ ($\frac{\infty}{\infty}$ form) or 0 ($\frac{0}{0}$ form).

Here, it's e^x so start over

$$L = \lim_{x \to \infty} \frac{\frac{1}{2}(e^x - e^{-x})}{\frac{1}{2}(e^x + e^{-x})} \cdot \frac{1/e^x}{1/e^x} \qquad \left(\text{or } \frac{e^{-x}}{e^{-x}}\right)$$

$$= \lim_{x \to \infty} \frac{\frac{1}{2}(1 - e^{-2x})}{\frac{1}{2}(1 + e^{-2x})} = \frac{\frac{1}{2}(1-0)}{\frac{1}{2}(1+0)} = 1 \quad \blacksquare$$

Ex 4 $\frac{0}{0}$ example – the derivative of $\sin x$

$$L = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

Here, h is the variable and x is a constant

top $\longrightarrow \sin(x+0) - \sin x = 0$

bottom $\longrightarrow 0$ so $\frac{0}{0}$

$$L \overset{L'H}{=} \lim_{h \to 0} \frac{\cos(x+h) - 0}{1} \qquad (\sin x = \text{constant})$$

$$= \frac{\cos(x+0)}{1} = \cos x$$