Section 4.4 part A: overview of determinate vs indeterminate forms

1. As a part of curve sketching, we want to know what happens to a function as $x \to \infty$ and $x \to -\infty$ (the far right and far left portions of the graph). So for this section, we are returning to a discussion of limits, but the discussion is not limited to just this graphical issue. Think of it as an important digression. Note also that we will not always look at limits as $x \to \infty$; sometimes it will be $x \to 0$ or $x \to 1$ for example. We will write all of these cases as $x \to a$ to simplify the notation. But when there is a specific problem, a will always be given.

   Make sure you study the last page of this document – there is a new tool called l'Hospital's Rule that is very powerful, that we use a lot. You'll like it. The guy's name is pronounced lope ee tol. Nothing to do with hospitals.

2. There are some basic limits that we use as building blocks in this section. Recall them now:

$$\begin{aligned}
\lim_{x\to\infty} e^x &= \infty \\
\lim_{x\to-\infty} e^x &= 0 \\
\lim_{x\to\infty} e^{-x} &= 0 \\
\lim_{x\to\infty} \ln x &= \infty \\
\lim_{x\to 0^+} \ln x &= -\infty \\
\lim_{x\to\infty} \frac{1}{x^p} &= 0 \\
\lim_{x\to\infty} \tan^{-1} x &= \frac{\pi}{2}
\end{aligned}$$

3. Pick 2 functions $f(x)$ and $g(x)$. We can combine them in a variety of ways:

$$\begin{aligned}
&f(x) + g(x) \\
&f(x) - g(x) \\
&f(x)g(x) \\
&\frac{f(x)}{g(x)} \\
&f(x)^{g(x)}
\end{aligned}$$

4. When we take the limit of any of these groupings, sometimes the limit is easy to identify and we don't need to do anything special. Such forms are called 'determinate'. A simple example with $f(x) = x^2$ and $g(x) = x^3$ is

$$\lim_{x\to 3} f(x) + g(x) = \lim_{x\to 3} x^2 + x^3 = 9 + 27 = 36$$

5. But sometimes, we get a form that is 'indeterminate'. This requires us to do some analysis in order to interpret the limit. Recall that we did some of this in Chapter 2 - we looked at limits of the form $\frac{0}{0}$ and used algebraic techniques to simplify. In Section 4.4, we are expanding that discussion to different forms.

   Notation: c and d represent finite constants that are not zero; we assume they are positive numbers. In the example above, $c = 9$ and $d = 27$. We single out 0 and 1 specifically because they often play a special role in the limit. And of course, infinity (∞) needs no introduction.

6. Forms for $\lim_{x \to a} f(x) \pm g(x)$. We can easily combine the plus and minus cases.

   (a) Determinate forms. For these cases, we can identify the final result without further analysis.

$$
\begin{array}{ll}
(a) & c + d = \text{a finite number} \\
(b) & c + \infty = \infty \\
(c) & \infty + \infty = \infty \\
(d) & c - \infty = -\infty \\
(e) & -\infty - \infty = -\infty
\end{array}
$$

   To interpret these, note that we are looking at the result of taking the limits of $f(x)$ and $g(x)$ individually and them combining them. For example, line (b) is a shorthand notation for $\lim_{x \to a} f(x) + g(x) = c + \infty = \infty$. Adding a finite number to infinity yields infinity. An example of this is $\lim_{x \to \infty} \tan^{-1} x + e^x = \frac{\pi}{2} + \infty = \infty$.
   A convenient way to think about these forms is that there is no conflict between $f(x)$ and $g(x)$. For example, in (c) both $f(x)$ and $g(x)$ want the limit to be infinity, so they are working together. In (e) they both want the limit to be negative infinity.

   (b) Indeterminate form for $\lim_{x \to a} f(x) \pm g(x)$. There is only one form that requires analysis:

$$\infty - \infty$$

   Here, $f(x)$ wants the limit to be plus infinity while $g(x)$ wants it to be negative infinity. There is a conflict! We will have to do some analysis to see which term wins. The final result could be negative infinity, positive infinity, zero or a negative or a positive number. See lecture notes part B for some examples.

7. Forms for $\lim_{x \to a} f(x) \cdot g(x)$.

(a) Determinate forms. For these cases, we can identify the final result without further analysis.

$$\begin{array}{rc} (a) & c \cdot d = \text{a finite number} \\ (b) & c \cdot \infty = \infty \\ (c) & \infty \cdot \infty = \infty \\ (d) & c \cdot (-\infty) = -\infty \\ (e) & (-\infty) \cdot (-\infty) = \infty \\ (f) & (-\infty) \cdot \infty = -\infty \end{array}$$

For each of these cases, there is no conflict between $f(x)$ and $g(x)$ so we know straight away what the limit is.

(b) Indeterminate form. There is only one case that requires analysis

$$0 \cdot \infty$$

One of the functions wants the limit to be 0 while the other wants the limit to be infinity. Which will win this titanic struggle? For any given problem, the final answer could be anything. See lecture notes part C for examples.

8. Forms for $\lim_{x\to a} \dfrac{f(x)}{g(x)}$. As before, if $f(x)$ and $g(x)$ are acting together, then the limit is determinate (requires no further analysis). But if they are in conflict, then further analysis will need to be done to identify the limit.

   A zero in the denominator wants the limit to be infinity. Infinity in the numerator wants the limit to be infinity.

   (a) Determinate forms. The cases are

   $$\begin{array}{ll} (a) & \frac{c}{d} = \text{a finite number} \\ (b) & \frac{c}{0} = \infty \\ (c) & \frac{\infty}{c} = \infty \\ (d) & \frac{\infty}{0} = \infty \\ (e) & \frac{c}{\infty} = 0 \end{array}$$

   Infinity dominates any constant, so its location in the numerator or denominator dictates the outcome. In (d), both numerator and denominator want the limit to be infinity so they are acting together.

   (b) Indeterminate forms. We looked at this a bit in Chapter 2. The forms that require manipulation to interpret are

   $$\frac{0}{0}, \qquad \frac{\infty}{\infty}$$

   Here, the numerator and denominator are in conflict, so the question is which one dominates the other. We will often use l'Hospital's Rule to resolve the conflict. See lecture notes part D for examples.

9. Forms for $\lim_{x \to a} f(x)^{g(x)}$. We haven't looked at limits of this exponential function form yet; these problems are the hardest in the section. The principles are that zero raised to a power is zero (unless the power is infinity or zero), 1 raised to any power is 1 (unless the power is infinity), and infinity raised to any power is infinity (unless the power is zero).

   (a) Determinate forms. Remember, c and d represent finite constants that are not zero.

$$\begin{aligned} c^d &= \text{a finite number} \\ 0^d &= 0 \\ 1^d &= 1 \\ c^\infty &= \infty \quad \text{if } c > 1 \\ c^\infty &= 0 \quad \text{if } 0 < c < 1 \\ c^0 &= 1 \\ \infty^d &= \infty \\ \infty^\infty &= \infty \end{aligned}$$

   (b) Indeterminate forms. In these forms, there is a conflict between the base and the exponent. A base that is zero wants the limit to be zero, a base that is 1 wants the limit to be 1, a base that is infinity wants the limit to be infinity. An exponent that is zero wants the limit to be 1, an exponent that is infinity wants the limit to be infinity or zero (depending on whether the base is bigger than 1 or smaller than 1). The forms are

$$\infty^0, \quad 1^\infty, \quad 0^0, \quad 0^\infty$$

   The way we will handle these is to use the identity $A = e^{\ln A}$. We can also notate this as $A = \exp(\ln A)$. The exp notation is handy when the argument (what's inside) is complicated. Recall also, that we can move a limit inside a function, so $\lim_{x \to a} e^{\ln A} = \exp\left(\lim_{x \to a} A\right)$. In this case, $A = f(x)^{g(x)}$, so the limit becomes

$$\begin{aligned} \lim_{x \to a} f(x)^{g(x)} &= \lim_{x \to a} e^{\ln(f(x)^{g(x)})} \\ &= \exp\left(\lim_{x \to a} \ln f(x)^{g(x)}\right) \\ &= \exp\left(\lim_{x \to a} g(x) \ln f(x)\right) \end{aligned}$$

   Notice that we turned the exponential form $f(x)^{g(x)}$ into a product, $g(x) \ln f(x)$, to make it easier to analyze. There are examples in lecture notes part E, and I introduce some notational aids so writing these bad boys on paper is easier.

10. l'Hospital's Rule. This powerful tool is the main result of the section. The rule states:

If $L = \lim_{x\to a} \dfrac{f(x)}{g(x)}$ has the form $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$, then the limit equals $\lim_{x\to a} \dfrac{f'(x)}{g'(x)}$. This is really useful if $f'(x)$ and $g'(x)$ are simpler than $f(x)$ and $g(x)$.

Some vital notes:

- This does NOT mean that $\dfrac{f(x)}{g(x)} = \dfrac{f'(x)}{g'(x)}$. It only means that both fractions have the same limit.
- Don't confuse this with the quotient rule. There is no connection.
- The limit must be in the form of a fraction. If we want to use this on the forms $\infty - \infty$ or $0 \cdot \infty$, for example, we need to convert them to a fraction first.
- I will abbreviate the rule as L'H. When we use it, I will write "apply L'H".
- L'H doesn't always work. Sometimes the second fraction $\dfrac{f'(x)}{g'(x)}$ is worse than the original $\dfrac{f(x)}{g(x)}$, and sometimes they are equivalent in difficulty.
- If the form of the original limit is not $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$, then applying L'H is not valid and will lead to wrong results.

Now go to lecture notes part B to get started on examples.