Exam 3 practice problems

1a

$f(x) = x^2 e^x - 15e^x - 7$

$0 = f'(x) = 2xe^x + x^2 e^x - 15e^x = (x^2+2x-15)e^x$

$= (x+5)(x-3)e^x \qquad$ CP : $x = -5, 3$

1b

$g(x) = \dfrac{x^2}{x-3}$

$g'(x) = \dfrac{(x-3)[2x] - x^2[1]}{(x-3)^2} = \dfrac{x^2 - 6x}{(x-3)^2} = \dfrac{x(x-6)}{(x-3)^2}$

CP : $x = 0, 6$ note : $x = 3$ is not a CP since it's not in the domain of f (VA there)

1c

$h(x) = x^{1/2}(x+1)$

$h'(x) = [\frac{1}{2}x^{-1/2}](x+1) + x^{1/2}[1] = \dfrac{\frac{1}{2}(x+1)}{\sqrt{x}} + \dfrac{\sqrt{x}}{1}$

$= \dfrac{\frac{1}{2}(x+1) + x}{\sqrt{x}} = \dfrac{\frac{3}{2}x + \frac{1}{2}}{\sqrt{x}}$ CP: $x = -\frac{1}{3}$, $x = 0$

2a

$f(x) = -\dfrac{x^3}{3} + \dfrac{x^2}{2} + 6x + 1 \qquad [0,4]$

$f'(x) = -x^2 + x + 6 = -(x^2 - x - 6) = -(x-3)(x+2)$

CP : $x = 3, -2$; only 3 is in $[0,4]$

candidates

$x = 0 \qquad f(0) = 1 \qquad \leftarrow$ abs min is 1

$x = 3 \qquad f(3) = 14.5 \qquad \leftarrow$ abs max is 14.5

$x = 4 \qquad f(4) = 11.67$

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2b $g(x) = \dfrac{x}{x^2 - x + 1}$ $[0,3]$

$g'(x) = \dfrac{(x^2-x+1)[1] - x[2x-1]}{(x^2-x+1)^2} = \dfrac{-x^2+1}{(x^2-x+1)^2}$

CP $x = \pm 1$; only $x=1$ is in $[0,3]$

candidates

$x=0$ $g(0) = 0$ ← abs min is 0

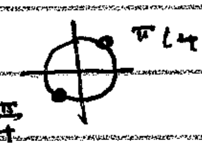
$x=1$ $g(1) = 1$ ← abs max is 1

$x=3$ $g(3) = .429$

2c $h(x) = e^x \cos x$ $[0,2\pi]$

$h'(x) = e^x \cos x - e^x \sin x = e^x(\cos x - \sin x)$

$h'(x) = 0$ where $\cos x = \sin x$



candidates

$x = 0$ $h(0) = 1$

$x = \pi/4$ $h(\frac{\pi}{4}) = e^{\pi/4}\cos\frac{\pi}{4} = 1.55$

$x = 5\pi/4$ $h(\frac{5\pi}{4}) = -35.89$ ← abs min is -35.89

$x = 2\pi$ $h(2\pi) = 535.49$ ← abs max is 535.49

3a $f(t)$ is Betsy's position.

$a = 0$ (1:00 pm) $f(a) = 120$ $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

$b = ?$ $f(b) = 210$ $66 = \dfrac{210-120}{b}$

$b = 1.36$

The time is 1.36 hours after 1:00pm.

.36 hours = .36 * 60 min = 21.6 min

The time is about 2:22 pm

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3b function is $T(t)$

$a = 0 \quad T(0) = 70 \qquad T'(c) = \frac{T(b) - T(a)}{b-a}$

$b = 5 \quad T(5) = 250 \qquad T'(c) = \frac{250-70}{5-0} = 36$

statement: The inst. ROC of temperature was 36 degrees/minute at some instant.

4a $f(x) = x^3 + 6x^2 + 9x - 1$

$f'(x) = 3x^2 + 12x + 9 \qquad$ CP $\quad x = -1, -3$

$f'' = 6x + 12 \qquad$ PIP $\quad x = -2$

	-3	-2	-1	
f'	$+$	$-$	$-$	$+$
f''	$-$	$-$	$+$	$+$

f is inc on $(-\infty, -3) \cup (-1, \infty)$ — Make sure you report the intervals

f is dec on $(-3, -1)$

f is CUP on $(-2, \infty)$

f is CAP on $(-\infty, -2)$

4b $g(x) = \frac{x^2}{x+1}$

$g'(x) = \frac{x^2+2x}{(x+1)^2} \qquad$ CP $\quad x = -2, 0$

$g''(x) = \frac{2}{(x+1)^3} \qquad$ PIP $\quad$ none $\quad$ (VA at $x=-1$ doesn't count)

	-2	-1 (VA)	0	
g'	$+$	$-$	$-$	$+$
g''	$-$	$-$	$+$	$+$

g is inc on $(-\infty, -2) \cup (0, \infty)$, dec on $(-2, -1) \cup (-1, 0)$

g is CUP on $(-1, \infty)$, CAP on $(-\infty, -1)$

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4c $h(x) = x\ln x$ $\boxed{\text{domain}: x > 0}$

$h'(x) = \ln x + 1$: CP : $\ln x = -1$ $x = e^{-1}$

$h''(x) = \frac{1}{x}$ PIP : none

	0	e^{-1}	
h'		$-$	$+$
h''		$+$	$+$

h is inc on (e^{-1}, ∞), dec on $(0, e^{-1})$

h is CUP on $(0, \infty)$

4d $r(x) = \ln(x^2+9)$

$r'(x) = \frac{2x}{x^2+9}$ CP : $x = 0$

$r''(x) = \frac{-2(x^2-9)}{(x^2+9)^2}$ PIP : $x = \pm 3$

	-3		0		3	
r'	$-$		$-$	$+$		$+$
r''	$-$		$+$	$+$		$-$

r is inc on $(0, \infty)$, dec on $(-\infty, 0)$

r is CUP on $(-3, 3)$, CAP on $(-\infty, -3) \cup (3, \infty)$

4e $\cancel{s(x) = e^x \sin x}$

$\cancel{s'(x) = e^x(\cos x + \sin x)}$

abandoned !

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5a $L = \lim_{x \to \infty} x^5 - x^4 \qquad \infty - \infty$

$= \lim_{x \to \infty} x^4(x-1) = \infty \cdot \infty = \infty$

5b $L = \lim_{x \to \infty} x - e^x \qquad \infty - \infty$

$= \lim_{x \to \infty} x\left(1 - \frac{e^x}{x}\right) \qquad \infty \cdot (1 - \text{what? examine separately})$

$\lim_{x \to \infty} \frac{e^x}{x} \overset{L'H}{=} \lim_{x \to \infty} \frac{e^x}{1} = \infty$

$\frac{\infty}{\infty}$

so back to the original

$L = \lim_{x \to \infty} x\left(1 - \frac{e^x}{x}\right) = \infty(-\infty) = -\infty$

5c $L = \lim_{x \to \infty} e^{-x} \tan^{-1} x = e^{-\infty} \cdot \frac{\pi}{2} = 0 \cdot \frac{\pi}{2} = 0$

5d $L = \lim_{x \to -\infty} x e^x \qquad (-\infty) \cdot e^{-\infty} = -\infty \cdot 0$

$= \lim_{x \to -\infty} \frac{x}{e^{-x}} \qquad \frac{-\infty}{e^{-(-\infty)}} = \frac{-\infty}{e^{\infty}} = -\frac{\infty}{\infty}$

$\overset{L'H}{=} \lim_{x \to -\infty} \frac{1}{-e^{-x}} = \lim_{x \to -\infty} -e^x = -e^{-\infty} = 0$

5e $L = \lim_{x \to \infty} \sqrt{x} \sin\left(\frac{1}{x}\right) \qquad \infty \cdot \sin\left(\frac{1}{\infty}\right) = \infty \cdot \sin(0) = \infty \cdot 0$

$= \lim_{x \to \infty} \frac{\sin\left(\frac{1}{x}\right)}{x^{-1/2}} \qquad \frac{\sin(0)}{1/\sqrt{\infty}} = \frac{0}{0}$

$\overset{L'H}{=} \lim_{x \to \infty} \frac{\cos\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)}{-\frac{1}{2}x^{-3/2}}$

I abandoned the original + changed the problem

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5e $L = \lim_{x \to \infty} x \sin\left(\frac{1}{x}\right)$ $\quad \infty \cdot \sin 0 = \infty \cdot 0$

$= \lim_{x \to \infty} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}}$ $\quad \frac{0}{1/\infty} = \frac{0}{0}$

$\overset{L'H}{=} \lim_{x \to \infty} \frac{\cos\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)}{-1/x^2} = \lim_{x \to \infty} \cos\left(\frac{1}{x}\right) = \cos 0 = 1$

5f $L = \lim_{x \to \infty} \frac{(\ln x)^3}{x^2}$ $\quad \frac{\infty}{\infty}$

$\overset{L'H}{=} \lim_{x \to \infty} \frac{3(\ln x)^2 \frac{1}{x}}{2x} = \lim_{x \to \infty} \frac{3(\ln x)^2}{2x^2}$ $\quad \frac{\infty}{\infty}$

$\overset{L'H}{=} \lim_{x \to \infty} \frac{6 \ln x \cdot \frac{1}{x}}{4x} = \lim_{x \to \infty} \frac{3 \ln x}{2x^2}$ $\quad \frac{\infty}{\infty}$

$\overset{L'H}{=} \lim_{x \to \infty} \frac{3 \frac{1}{x}}{4x} = \lim_{x \to \infty} \frac{3}{4x^2} = 0$

5g $L = \lim_{x \to 1} \frac{x \ln x}{e^{x-1} - 1}$ $\quad \frac{1 \cdot 0}{e^0 - 1} = \frac{0}{0}$

$\overset{L'H}{=} \lim_{x \to 1} \frac{\ln x + x\left(\frac{1}{x}\right)}{e^{x-1}} = \frac{\ln 1 + 1}{e^0} = 1$

5h $L = \lim_{x \to 0} \frac{\sin 5x}{\tan 6x}$ $\quad \frac{0}{0}$

$\overset{L'H}{=} \lim_{x \to 0} \frac{5 \cos 5x}{6 \sec^2 6x} = \frac{5 \cos 0}{6 \sec^2 0} = \frac{5}{6}$

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5i $L = \lim_{x\to\infty} \frac{x^3 \ln x}{e^x}$ $\frac{\infty}{\infty}$

$\overset{L'H}{=} \lim_{x\to\infty} \frac{3x^2\ln x + x^2}{e^x}$ $\frac{\infty}{\infty}$

$\overset{L'H}{=} \lim_{x\to\infty} \frac{(6x\ln x + 3x) + 2x}{e^x}$ $\frac{\infty}{\infty}$ $\frac{6x\ln x + 5x}{e^x}$

$\overset{L'H}{=} \lim_{x\to\infty} \frac{(6\ln x + 6) + 5}{e^x}$ $\frac{\infty}{\infty}$

$\overset{L'H}{=} \lim_{x\to\infty} \frac{6/x}{e^x} = \frac{0}{\infty} = 0$

5j $L = \lim_{x\to 0} x^{1/x} = \lim_{x\to 0} e^{\ln x^{1/x}} = \exp\left(\lim_{x\to 0} \frac{1}{x}\ln x\right)$

$L = e^k$ $k = \lim_{x\to 0} \frac{1}{x}\ln x = \infty\cdot(-\infty) = -\infty$

$L = e^{-\infty} = 0$

5k $L = \lim_{x\to\infty} \left(1+\frac{3}{x}\right)^{4x} = \lim_{x\to\infty} e^{\ln(1+\frac{3}{x})^{4x}}$

$= \exp\left(\lim_{x\to\infty} 4x\ln\left(1+\frac{3}{x}\right)\right)$

$= e^k$

$k = \lim_{x\to\infty} 4x\ln\left(1+\frac{3}{x}\right)$

$= \lim_{x\to\infty} \frac{4\ln\left(1+\frac{3}{x}\right)}{1/x}$ keep the 4 up top of $\frac{\ln(\;)}{1/(4x)}$ instead

$\frac{4\cdot\ln(1+0)}{1/\infty} = \frac{0}{0}$

~~$k \overset{L'H}{=} \lim_{x\to\infty}$~~

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5k $K = \lim_{x\to\infty} \dfrac{4\ln(1+\frac{3}{x})}{1/x}$ $\quad \frac{0}{0}$

$\overset{L'H}{=} \lim_{x\to\infty} \dfrac{4\ \frac{1}{1+\frac{3}{x}}\cdot(-\frac{3}{x^2})}{-\frac{1}{x^2}}$

$= \lim_{x\to\infty} 4\cdot\frac{1}{1+3/x}\cdot 3 = 12$

so $L = \lim_{x\to\infty}(1+\frac{3}{x})^{4x} = e^K = e^{12}$

5L $L = \lim_{x\to 0}(1+5x)^{2/x} = \lim_{x\to 0} e^{\ln(1+5x)^{2/x}}$

$= \exp\left(\lim_{x\to 0} \frac{2}{x}\ln(1+5x)\right)$

$= e^K$

$K = \lim_{x\to 0} \dfrac{2\ln(1+5x)}{x}$ $\quad \frac{0}{0}$

$\overset{L'H}{=} \lim_{x\to 0} \dfrac{2\cdot\frac{1}{1+5x}\cdot 5}{1} = 10$

so $L = e^K = e^{10}$

5m ~~$L = \lim_{x\to 0}(\sin x)^x = \lim_{x\to 0} e^{\ln(\sin x)^x}$~~

~~$= \exp\left(\lim_{x\to 0} x\ln(\sin x)\right)$~~

~~$= e^K$~~

~~$K = \lim_{x\to 0} x\ln(\sin x)$~~ $\quad 0\cdot\ln(0) = 0\cdot(-\infty)$

~~$= \lim_{x\to 0} \dfrac{\ln(\sin x)}{1/x}$~~ $\quad \frac{-\infty}{\infty}$

~~$\overset{L'H}{=} \lim_{x\to 0} \dfrac{\frac{1}{\sin x}\cdot\cos x}{-1/x^2} = \lim_{x\to 0}\dfrac{\cot x}{-1/x^2}$~~ $\quad \frac{\infty}{-\infty}$

ok, this is going nowhere. Let's abandon it.

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5m $L = \lim_{x\to 0} \left(\frac{1}{x}\right)^x$ ∞^0

$= \lim_{x\to 0} e^{\ln\left(\frac{1}{x}\right)^x} = \exp\left(\lim_{x\to 0} x\ln\left(\frac{1}{x}\right)\right)$

$0\cdot\ln\infty = 0\cdot\infty$

$= e^k$

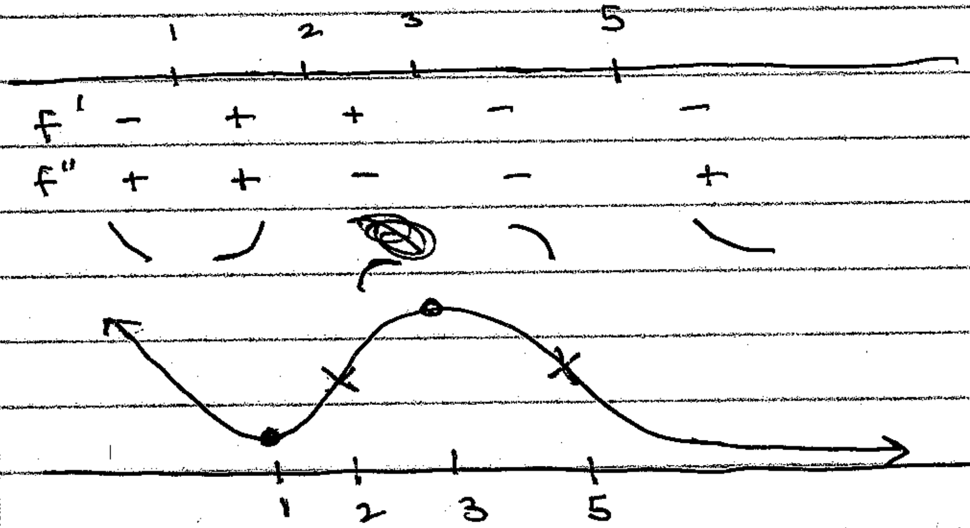
$k = \lim_{x\to 0} x\ln\left(\frac{1}{x}\right) = \lim_{x\to 0} -x\ln x$ still $0\cdot\infty$

$= \lim_{x\to 0} -\frac{\ln x}{1/x}$ $\frac{\infty}{\infty}$

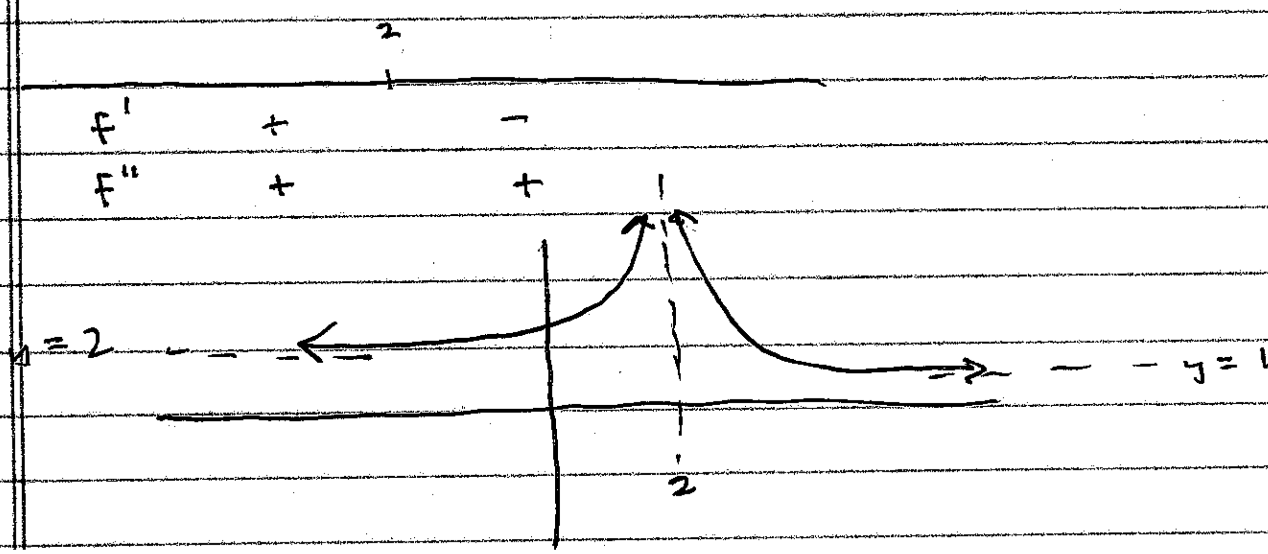
$\overset{L'H}{=} \lim_{x\to 0} \frac{1/x}{-1/x^2} = \lim_{x\to 0} -x = 0$

so $L = \lim_{x\to 0} \left(\frac{1}{x}\right)^x = e^k = e^0 = 1$

6a



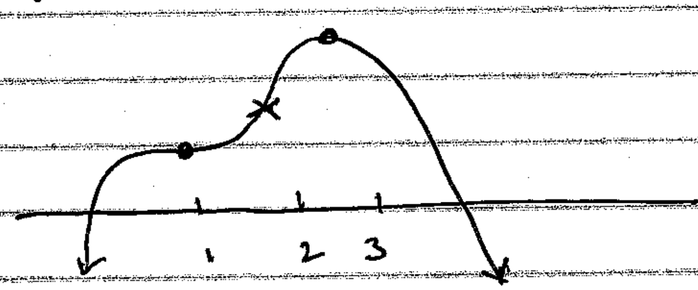
6b



Pg 10

6c

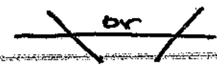
	1		2		3	
f'	$+$		$+$		$+$	$-$
f''	$-$		$+$		$-$	$-$

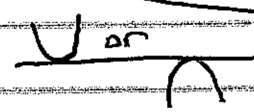


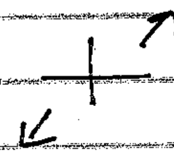
7a

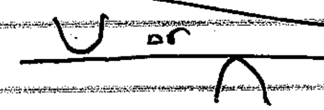
$f(x) = x(x+1)^2(x-2)^2$

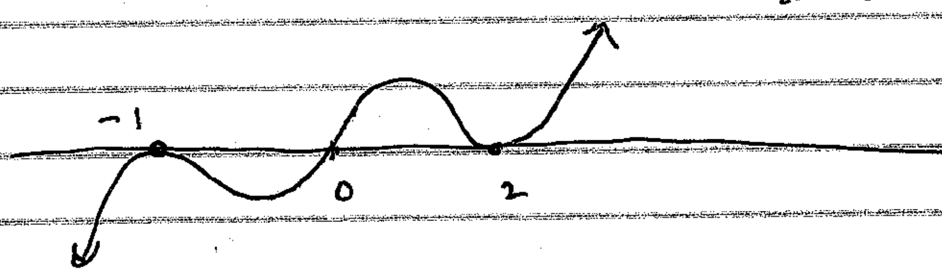
roots: $x = 0, -1, 2$

$x = 0$ 

$x = -1$ 

$f(x) \approx x \cdot x^2 \cdot x^2 = x^5$ 

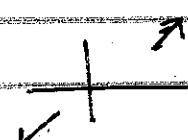
$x = 2$ 

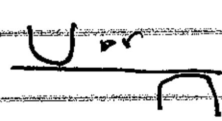


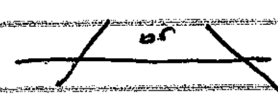
7b

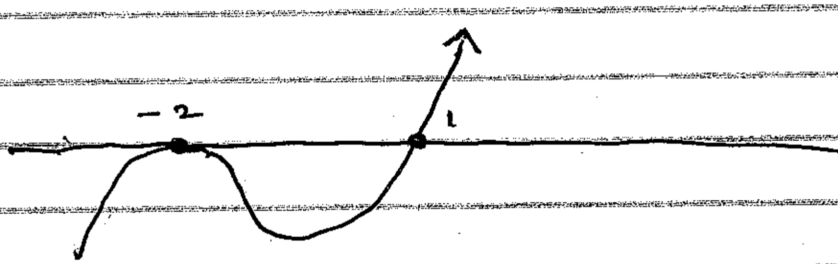
$f(x) = (x+2)^4(x-1)$

roots: $x = -2, 1$

$f(x) \approx x^4 \cdot x = x^5$ 

$x = -2$ 

$x = 1$ 



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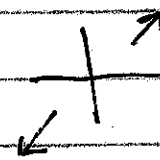
7c $f(x) = x^3(x-2)(x-3)$

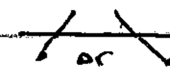
roots $x = 0, 2, 3$ $f(x) \approx x^3 x x = x^5$

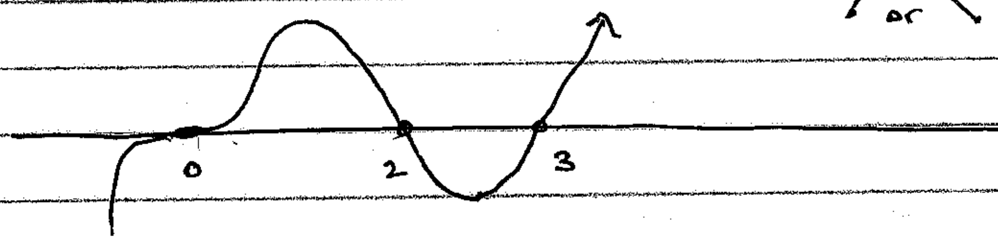


$x = 0$

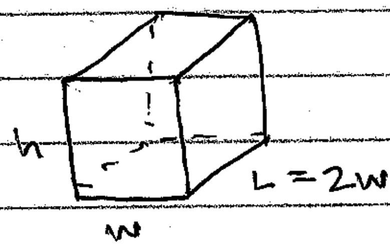
$x = 2$

$x = 3$





8a



$V = hwL = hw(2w)$
$= 2hw^2$

area = 4 sides + bottom

$A = 2(hw) + 2(hL) + Lw$

front, back left, right

$A = 2hw + 4hw + 2w^2 = 10000 \text{ cm}^2$

$6hw + 2w^2 = 10000$

$6hw = 10000 - 2w^2$

$h = \frac{10000 - 2w^2}{6w} = \frac{10000}{6} \cdot \frac{1}{w} - \frac{1}{3} w$

so

$V = 2hw^2 = 2\left[\frac{10000}{6} \cdot \frac{1}{w} - \frac{1}{3} w\right] w^2 = \frac{10000}{3} w - \frac{2}{3} w^3$

$\frac{dV}{dw} = 0 = \frac{10000}{3} - 2w^2$ so $w^2 = \frac{10000}{6} = 1666.7$

$w = 40.8$ cm

$L = 2w = 81.6$ cm

$h = \frac{10000}{6w} - \frac{1}{3} w = 27.2$ cm

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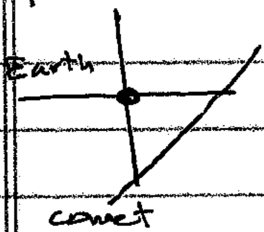
8b



$3x - 4y = 9$, $\quad 4y = 3x - 9$

$$y = \frac{3}{4}x - \frac{9}{4}$$

distance between $(0,0)$ and any point on path is

$$D^2 = (x-0)^2 + (y-0)^2 = x^2 + \left(\frac{3}{4}x - \frac{9}{4}\right)^2$$

call it F

$$F'(x) = 2x + 2\left(\frac{3}{4}x - \frac{9}{4}\right)\cdot\frac{3}{4}$$

$$= 2x + \frac{9}{8}x - \frac{27}{8} = \frac{25}{8}x - \frac{27}{8}$$

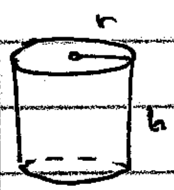
• set $F'(x) = 0$ to get $x = 27/25$

then $y = \frac{3}{4}\cdot\frac{27}{25} - \frac{9}{4} = -1.44$

then $D^2 = \left(\frac{27}{25}\right)^2 + (-1.44)^2 = 3.24$

The distance is $D = \sqrt{3.24} = 1.8$ million miles

8c



$$V = \pi r^2 h$$

$$A = 2\pi r h + 2\pi r^2$$

top + bottom are circles

flatten the side: h $\boxed{}$ $2\pi r$ circumference of circle

$$500 = A = 2\pi r h + 2\pi r^2$$

$$500 - 2\pi r^2 = 2\pi r h$$

$$\frac{500 - 2\pi r^2}{2\pi r} = h$$

$$h = \frac{250}{\pi r} - r$$

so $V = \pi r^2\left(\frac{250}{\pi r} - r\right) = 250 r - \pi r^3$

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8c $V = 250r - \pi r^3$

~~set~~ $\frac{dV}{dr} = 0 = 250 - 3\pi r^2$

so

$r^2 = \frac{250}{3\pi}$

$r = 5.15$ cm

$h = \frac{250}{\pi r} - r = 10.3$ cm