1. Find the local maximum and minimum values of \( f(x) = \frac{x}{x^2 + 4} \) using both the First and Second Derivative Tests. Which method do you prefer?

Solution: We have \( f'(x) = \frac{1(x^2 + 4) - x \cdot 2x}{(x^2 + 4)^2} = \frac{4 - x^2}{(x^2 + 4)^2} \).

Thus, the critical numbers are where \( 4 - x^2 = 0 \), or \( x = \pm 2 \).

(a) For the First Derivative Test, we need only look at the sign of \( f'(x) \):

\( f'(x) < 0 \) if \( x < -2 \) or \( 2 < x \), \( f'(x) > 0 \) if \( -2 < x < 2 \).

So, \( x = -2 \) is a local minimum and \( x = 2 \) is a local maximum.

(b) For the Second Derivative Test, we have that

\[
f''(x) = \frac{-2x(x^2 + 4)^2 - (4 - x^2) \cdot 2(x^2 + 4) \cdot 2x}{(x^2 + 4)^4} = \frac{2x(x^2 - 12)}{(x^2 + 4)^3}.
\]

Then \( f''(-2) = (-4)(-8)/(4 + 4)^3 > 0 \), so that \( x = -2 \) is a local minimum, and \( f''(2) = (+4)(-8)/(4 + 4)^3 < 0 \), so that \( x = 2 \) is a local maximum.

2. Sketch the graph of a function which satisfies all the following conditions:

(a) \( f'(1) = f'(-1) = 0, f'(x) < 0 \) if \( |x| < 1 \)

(b) \( f'(x) > 0 \) if \( 1 < |x| < 2 \), \( f'(x) = -1 \) if \( |x| > 2 \)

(c) \( f''(x) < 0 \) if \( -2 < x < 0 \), inflection point \((0,1)\)

Solution:

3. Sketch the graph of a function which satisfies all the following conditions:

(a) \( f(0) = f'(0) = f'(2) = f'(4) = f'(6) = 0 \)

(b) \( f'(x) > 0 \) if \( 0 < x < 2 \) or \( 4 < x < 6 \)
(c) \( f'(x) < 0 \) if \( 2 < x < 4 \) or \( 6 < x \)
(d) \( f''(x) > 0 \) if \( 0 < x < 1 \) or \( 3 < x < 5 \)
(e) \( f''(x) < 0 \) if \( 1 < x < 3 \) or \( 5 < x \)
(f) \( f(-x) = f(x) \) for all \( x \)

**Solution:**

![Graph of f'(x) showing increasing and decreasing intervals, local maxima and minima, concavity, and points of inflection.]

4. Using the graph of \( f'(x) \) shown:

(a) Find the intervals on which \( f(x) \) is increasing, and those where it is decreasing.
(b) Find the local maxima and minima of \( f(x) \).
(c) Find the intervals where \( f(x) \) is concave up, and where it is concave down.
(d) Find the \( x \)-coordinate(s) of any points of inflection.
(e) Assuming that \( f(0) = 0 \), sketch the graph of \( f(x) \).
(a) $f(x)$ is increasing where $f'(x) > 0$, so $(1, 6) \cup (8, 9)$, and decreasing where $f'(x) < 0$, so $(0, 1) \cup (6, 8)$.

(b) By the First Derivative Test, the minima are $x = 1$ and $x = 8$, and the maximum is $x = 6$.

(c) $f(x)$ is concave up where $f'(x)$ is increasing, so $(0, 2) \cup (3, 5) \cup (7, 9)$ and concave down where $f'(x)$ is decreasing, so $(2, 3) \cup (5, 7)$.

(d) The inflection points are where the concavity changes. These occur at $x = 2$, $x = 3$, $x = 5$ and $x = 7$.

(e) The graph is:

5. For the function $f(x) = 3x^{2/3} - x$, find the intervals where $f(x)$ is increasing, where it is decreasing, where it is concave up, and where it is concave down. Also identify the local extrema and inflection points.

Use all of this information to sketch the graph.

**Solution:** The domain of $f(x)$ is all real $x$.

We have $f'(x) = 3(2/3)x^{-1/3} - 1 = 2x^{-1/3} - 1$.

Thus, the critical numbers are $x = 0$ (the derivative is undefined) and $x = 8$, since $f'(8) = 0$.

For $x < 0$ and $8 < x$, $f'(x) < 0$, so it is decreasing there.

For $0 < x < 8$, $f'(x) > 0$, so $f(x)$ is increasing there.

By the First Derivative Test, $x = 0$ is a local minimum, $x = 8$ is a local maximum.

We also have that $f''(x) = -\frac{2}{3}x^{-4/3}$. Thus, $f''(x) < 0$ if $x \neq 0$. So, the graph is concave down everywhere except $x = 0$, and has no inflection points.