1. Use differentials to estimate the amount of paint required to put a coat of paint 0.05 cm thick on a hemispherical dome of diameter 50 m.

Solution: The amount of paint required will be the difference in volume when the radius changes by 0.05 cm.

The volume of a hemisphere of radius $r$ is $V = \frac{2}{3}\pi r^3$.

We have $r = 2500$ cm, and $dr = 0.05$ cm.

The approximate change is $\Delta V \approx dV = 2\pi r^2 dr = 2\pi(2500)^2(0.05) \approx 1,963,495$ cm$^3 \approx 1,963$ litres.

2. Show that $\cosh(2x) = 2 \cosh^2(x) - 1$.

Solution: We have

$$2 \cosh^2(x) - 1 = 2 \left( \frac{e^x + e^{-x}}{2} \right)^2 - 1 = 2 \left( \frac{e^{2x} + 2 + e^{-2x}}{4} \right) - 1 = \frac{e^{2x} + e^{-2x}}{2} = \cosh(2x)$$

3. Given $y = \frac{\sinh(2x^2)}{1 + \cosh(2x^2)}$, find where the tangent line is horizontal.

Solution: Differentiating, $y' = \frac{\cosh(2x^2) \cdot 4x(1 + \cosh(2x^2)) - \sinh(2x^2) \cdot \sinh(2x^2) \cdot 4x}{(1 + \cosh(2x^2))^2}$

$$= \frac{4x(\cosh(2x^2) + \cosh^2(2x^2) - \sinh^2(2x^2))}{(1 + \cosh(2x^2))^2}$$

$$= \frac{4x(\cosh(2x^2) + 1)}{(1 + \cosh(2x^2))^2} = \frac{4x}{1 + \cosh(2x^2)}$$

Thus, the only value for which $y' = 0$ is $x = 0$.

4. Find the absolute maximum and absolute minimum of $f(x) = \frac{x^3}{27 + x^4}$ on the interval [-1,4].

Solution: $f(x)$ is defined for all $x$. Taking derivatives,

$$y = \frac{3x^2(27 + x^4) - x^3 \cdot 4x^3}{(27 + x^4)^2} = \frac{x^2(81 + 3x^4 - 4x^4)}{(27 + x^4)^2} = \frac{x^2(81 - x^4)}{(27 + x^4)^2}$$

Thus, the critical numbers are $x = 0, -3, 3$. We ignore -3, since it is not in the interval.

We have $f(-1) = -1/28, f(0) = 0, f(3) = 27/(27+81) = 1/4$ and $f(4) = 64/(27+256) < 1/4$. So, the absolute maximum is $x = 3$ and absolute minimum is $x = -1$. 

5. Show that there is a unique solution to $e^{-x} = 3x$

Solution: Let $f(x) = e^{-x} - 3x$. The given equation is equivalent to $f(x) = 0$. Now, $f(x)$ is continuous for all $x$, and $f(-1) = e + 3 > 0$, $f(1) = e^{-1} - 3 < 0$.

By the Intermediate Value Theorem, there must exist at least one value $-1 < c < 1$ for which $f(c) = 0$.

Suppose that there were two solutions, say $c_1 < c_2$ with $f(c_1) = 0 = f(c_2)$.

Since $f(x)$ is differentiable for all $x$, Rolle’s Theorem would mean that there is a value $c_1 < b < c_2$ such that $f'(b) = 0$.

But, $f'(x) = -e^{-x} - 3 < -3$, so no such $b$ exists. Hence, the solution must be unique.