Homework #4  Calculus I 3450:221  Dr. Norfolk

1. Suppose that \( f(2) = -3, g(2) = 4, f'(2) = -2 \) and \( g'(2) = 7 \).

Find \( h'(2) \) for the following:

(a) \( h(x) = 5f(x) - 4g(x) \)
(b) \( h(x) = f(x)g(x) \)
(c) \( h(x) = \frac{f(x)}{g(x)} \)
(d) \( h(x) = \frac{f(x)}{1 + f(x)} \)

Solution:

(a) \( h'(2) = 5f'(2) - 4g'(2) = 5(-2) - 4(7) = -38 \)
(b) \( h'(2) = f'(2)g(2) + f(2)g'(2) = (-2)(4) + (-3)(7) = -29 \)
(c) \( h'(2) = \frac{f'(2)g(2) - f(2)g'(2)}{[g(2)]^2} = \frac{(-2)(4) - (-3)(7)}{4^2} = \frac{13}{16} \)
(d) \( h'(2) = \frac{f'(2)[1 + f(2)] - f(2)f'(2)}{[1 + f(2)]^2} = -\frac{2}{(1 - 3)^2} = -\frac{1}{2} \)

2. Let \( f(x) = \begin{cases} x^2 & \text{if } x \leq 2 \\ mx + b & \text{if } 2 < x \end{cases} \). Find the values of \( m \) and \( b \) that make \( f(x) \) differentiable everywhere.

Solution:

(a) Clearly, \( f'(x) = 2x \) for \( x < 2 \), \( f'(x) = m \) for \( x > 2 \).

(b) If we want \( f(x) \) differentiable at \( x = 2 \), it must be continuous there.

We have

\[
\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} x^2 = 4; \quad f(2) = 2^2 = 4; \quad \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (mx + b) = 2m + b.
\]

So, we need \( 2m + b = 4 \).

(c) To find the derivative (if it exists) at \( x = 2 \), we need to look at the one-sided limits.

\[
\lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^-} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2^-} (x + 2) = 4.
\]

\[
\lim_{x \to 2^+} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^+} \frac{mx + b - 4}{x - 2} = \lim_{x \to 2^+} \frac{mx + (4 - 2m) - 4}{x - 2} = \lim_{x \to 2^+} \frac{mx + (4 - 2m) - 4}{x - 2} = \lim_{x \to 2^+} \frac{mx}{x - 2} = m.
\]

So \( m = 4 \) and thus \( b = -4 \).
3. Given \( f(x) = xe^x + \sec x \), find \( f''(0) \).

\[
\begin{align*}
f(x) &= xe^x + \sec x, \\
f'(x) &= e^x + xe^x + \sec x \tan x, \\
f''(x) &= e^x + e^x + xe^x + \sec x \tan^2 x + \sec^3 x.
\end{align*}
\]

So, \( f''(0) = 1 + 1 + 0 + 0 + 1 = 3. \)

4. Find the points on the curve \( y = \frac{\cos x}{2 + \sin x} \) at which the tangent line is horizontal.

**Solution:** We have \( y' = \frac{-\sin x(2 + \sin x) - \cos x(\cos x)}{(2 + \sin x)^2} = \frac{-2\sin x - \sin^2 x - \cos^2 x}{(2 + \sin x)^2} \)

\[
= -1 - 2\sin x \quad \text{since } \sin^2 x + \cos^2 x = 1.
\]

The tangent line is horizontal iff its slope is 0. Solving \( y' = 0 \) means that \( \sin x = -\frac{1}{2} \), which gives \( x = \frac{7\pi}{6} + 2\pi n, \frac{11\pi}{6} + 2\pi n \), where \( n \) is any integer.

5. Find \( \lim_{x \to 0} \frac{\sin 4x}{\sin 6x} \).

**Solution:**

\[
\lim_{x \to 0} \frac{\sin 4x}{\sin 6x} = \lim_{x \to 0} \frac{\sin 4x}{4x} \cdot \frac{6x}{\sin 6x} \cdot \frac{4x}{6x} = \lim_{x \to 0} \frac{\sin 4x}{4x} \cdot \lim_{x \to 0} \frac{6x}{\sin 6x} \cdot \lim_{x \to 0} \frac{4x}{6x} = 1 \cdot 1 \cdot \frac{4}{6} = \frac{2}{3}.
\]