Here is a review of the integration and trigonometric skills you need in this course.

**Integration by substitution**

\[ \int e^{2x} \, dx = \frac{1}{2} e^{2x} + c \quad u = 2x, \, du = 2 \, dx \]

\[ \int 2x^3 \sqrt{1 + x^2} \, dx = \int (u - 1)^{1/2} \, du = \int u^{3/2} - u^{1/2} \, du \quad u = 1 + x^2, \, du = 2x \, dx, \, x^2 = u - 1 \]

**Integration by parts**

There are 3 classes of integrals that can be evaluated using IBP:

A) \( \int (\text{polynomial})(\sin \text{ or } \cos) \, dx \), \( \int (\text{polynomial})(\exp) \, dx \). Let \( u = \text{polynomial} \) to kill off the polynomial one degree at a time.

B) \( \int (\text{polynomial})(\logarithm) \, dx \), \( \int \arctrig \, dx \). Let \( u = \logarithm \) or \( u = \arctrig \).

C) \( \int (\exp)(\sin \text{ or } \cos) \, dx \). IBP twice, using \( u = \arctrig \) both times, or \( u = \exp \) both times. Called FOLDING.

**Basic Trigonometric Skills**

A) Definitions of the 6 trig functions in terms of right triangles and in terms of sines and cosines.

B) Pythagorean identities, 3 flavors. \( \sin^2 x + \cos^2 x = 1 \), \( 1 + \cot^2 x = \csc^2 x \), \( \tan^2 x + 1 = \sec^2 x \).

C) Half angle identities. \( \sin^2 \frac{x}{2} = \frac{1}{2} (1 - \cos x) \), \( \cos^2 \frac{x}{2} = \frac{1}{2} (1 + \cos x) \).

D) Double angle identities. \( 2 \sin x \cos x = \sin 2x \), \( \cos 2x - \sin 2x = \cos (2x) \), \( 2 \cos^2 x - 1 = \cos (2x) \), \( 1 - 2 \sin^2 x = \cos (2x) \).

E) Sum and difference identities. \( \sin(x \pm y) = \sin x \cos y \pm \sin y \cos x \), \( \cos(x \pm y) = \cos x \cos y \mp \sin x \sin y \).

F) Product identities. \( \sin A \cos B = \frac{1}{2}(\sin(A - B) + \sin(A + B)) \), \( \sin A \sin B = \frac{1}{2}(\cos(A - B) - \cos(A + B)) \), \( \cos A \cos B = \frac{1}{2}(\cos(A - B) + \cos(A + B)) \).

G) Elementary antiderivatives. \( \int \sin x \, dx = - \cos x + c \), \( \int \cos x \, dx = \sin x + c \), \( \int \sec^2 x \, dx = \tan x + c \), \( \int \csc^2 x \, dx = - \cot x + c \), \( \int \sec x \tan x \, dx = \sec x + c \), \( \int \sec x \, dx = \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x} \, dx = \int \frac{1}{u} \, du = \ln |\sec x + \tan x| + c \).

**Trigonometric Integrals**

A) \( \int \sin^n x \cos^m x \, dx \). If \( n \) or \( m \) is odd, pull off the odd power of sin or cos to use in \( du \), let \( u \) equal the other trig function, and apply Pythagorean identities if needed. If both \( n \) and \( m \) are odd, you can let \( u \) equal \( \sin x \) or \( \cos x \). If \( n \) and \( m \) are both even, apply half angle identities and Pythagorean identities.
Examples:

\[
\int \cos^3 x \, dx = \int (1 - \sin^2 x)(\cos x \, dx) = \int (1 - u^2) \, du
\]

\[
\int \sin^3 x \cos^3 x \, dx = \int (\sin^3 x)(1 - \sin^2 x)(\cos x \, dx) = \int (u^3)(1 - u^2)(\, du)
\]

\[
\int \sin^2 x \, dx = \int \frac{1}{2}(1 - \cos(2x)) \, dx
\]

B) \( \int \tan^m x \sec^n x \, dx \). If \( n \) is even, pull off a \( \sec^2 x \) to use in \( du \), let \( u = \tan x \) and apply Pythagorean identity if needed. If \( m \) is odd, pull off a \( \sec x \tan x \) to use in \( du \), let \( u = \sec x \), apply Pythagorean identity if needed. If both \( n \) is even and \( m \) is odd, you can do either of the above. If \( n \) is odd and \( m \) is even, God help you. Use Pythagorean identities to reduce to nothing but powers of \( \sec x \). Integrals of \( \sec x \) and \( \sec^3 x \) are standard, and can be looked up in your old Calculus text (you didn’t sell it back to the Bookstore, did you?). Higher odd powers of \( \sec x \) require careful use of IBP. Even powers can be handled by the second sentence in this paragraph.

Examples:

\[
\int \tan^2 x \sec^4 x \, dx = \int \tan^2 x(\tan^2 x - 1)(\sec^2 x \, dx) = \int u^2(u^2 - 1) \, du
\]

\[
\int \tan x \sec^{20} x \, dx = \int \sec^{19} x(\sec x \tan x \, dx) = \int u^{19} \, du
\]

\[
\int \tan^2 x \sec x \, dx = \int (\sec^2 x - 1) \sec x \, dx = \int \sec^3 x - \sec x \, dx
\]

Trigonometric Substitution

Look for expressions of the form \( \sqrt{a^2 - x^2} \), \( \sqrt{x^2 - a^2} \), or \( \sqrt{x^2 + a^2} \), or one of these to a higher power (for example, \((x^2 - 4)^2\) is the square root raised to the fourth power). If there is a minus under the square root, the positive term is the square of the hypotenuse and the negative term is the square of one of the legs; the square root is the other leg. If there is a plus under the square root, then the square root is the hypotenuse, and the terms inside are the squares of the legs. Use the triangle to make a substitution, which converts the integral to a trig integral (see above).

Examples:

1) \( \int \frac{1}{x^2\sqrt{16 - x^2}} \, dx \). The hypotenuse is 4 and the sides are \( x \) and \( \sqrt{16 - x^2} \). Let \( x = 4 \sin \theta \), so \( dx = 4 \cos \theta \). The integral becomes \( \int \frac{1}{16 \sin^2 \theta \sqrt{16 - 16 \sin^2 \theta}} \cdot 4 \cos \theta \, d\theta = \frac{1}{16} \int 1 \sin^2 \theta \, d\theta = \frac{1}{16} \cot \theta + c = \frac{1}{16} \frac{\sqrt{16 - x^2}}{x} + c \).

2) \( \int \frac{x^3}{\sqrt{9 + x^2}} \, dx \). The hypotenuse is \( \sqrt{9 + x^2} \) and the legs are \( x \) and 3. Let \( x = 3 \tan \theta \), so \( dx = 3 \sec^2 \theta \, d\theta \). The integral becomes \( \int \frac{27 \tan^3 \theta}{3 \sec \theta} \cdot 3 \sec^2 \theta \, d\theta = 27 \int \frac{\sin^3 \theta}{\cos^4 \theta} \, d\theta \). Write the numerator as \( \sin^2 \theta \sin \theta \, d\theta = (1 - u^2) \, du \) for \( u = \cos \theta \). The integral becomes \( 27 \int (1 - u^2)u^{-4} \, (-du) = 27 \left( -\frac{1}{u} + \frac{1}{3u^3} \right) = \left( -9\sqrt{9 + x^2} + \frac{1}{3}(9 + x^2)^{3/2} \right) \). Note that \( u = \cos \theta = 3/\sqrt{9 + x^2} \). Oh, and don’t forget the constant of integration.