1. Determine whether or not \( \lim_{{(x,y) \to (0,0)}} \frac{x^2 + 5xy}{x^2 + y^2} \) exists. If it exists, determine its value.

Let \( \gamma = \frac{y}{x} \)

\[
L = \lim_{{x \to 0}} \frac{x^2 + 5xy}{x^2 + y^2} = \lim_{{x \to 0}} \frac{x^2 (1 + 5\gamma)}{x^2 (1 + \gamma^2)} = \frac{1 + 5\gamma}{1 + \gamma^2}
\]

different paths yield different limits, so limit DNE

2. Find the partial derivatives \( f_x, f_y \) and \( f_{xy} \) for \( f(x, y) = \arctan (x^2 y) \). Simplify the numerators.

\[
\left( \tan^{-1} u \right)' = \frac{u'}{1 + u^2}
\]

\[
f_x = \frac{2xy}{1 + x^4 y^2}
\]

\[
f_y = \frac{x^2}{1 + x^4 y^2}
\]

\[
f_{xy} = \frac{(1 + x^4 y^2) \left[ 2x \right] - (2xy) \left[ 2x^3 y \right]}{(1 + x^4 y^2)^2} = \frac{2x - 2x^3 y^2}{(1 + x^4 y^2)^2}
\]
3. Use a linear approximation to estimate \( Q = \frac{.97^3}{3 + 1.01^{15}} \).

\[
F(x, y) = \frac{x}{3+y^{15}} \quad \text{ref \: \:} x = 1, \: y = 1
\]

\[
Q \approx F(1,1) + F_x(1,1)(x-1) + F_y(1,1)(y-1)
\]

\[
\begin{align*}
F(1,1) & = \frac{1}{4} \\
F_x & = \frac{3x^2}{3+y^{15}} \quad F_x(1,1) = \frac{3}{3+1} = \frac{3}{4} \\
F_y & = \frac{-x^3y^{14}}{(3+y^{15})^2} \quad F_y(1,1) = \frac{-15}{16}
\end{align*}
\]

\[
Q \approx \frac{1}{4} + \frac{3}{4}(0.03) - \frac{15}{16}(0.01)
\]

\[= \: 0.218\]

4. A computer’s CPU is a rectangular box with a square base (side length \(x\)) and height \(y\). The volume of the CPU is \( V = x^2y \). The ideal dimensions should be \( x = 3 \) cm and \( y = 0.4 \) cm. During manufacturing, however, the actual dimensions are \( x = 3 \pm 0.02 \) cm and \( y = 0.4 \pm 0.005 \) cm. Use differentials to estimate the maximum possible error in the volume of the manufactured CPUs.

\[
\begin{align*}
\text{Error} & = 12x \pm 0.02 \\
& = 12y \pm 0.005
\end{align*}
\]

\[
V = x^2y = 3.16
\]

\[
dV = (2xy) \: dx + (x^2) \: dy = 2.4 \: dx + 9 \: dy
\]

\[
|dV| \leq (2.4)(0.02) + 9(0.005)
\]

\[= 0.093\]
5. Let \( f(x, y) = x^2 + y^2 + y \). Use the chain rule to find the angular derivative in polar coordinates, \( \frac{\partial f}{\partial \theta} \). Simplify your answer and write it in polar coordinates.

\[
\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \theta}
\]

\[
= \left[ 2x \right] \left[ -r \sin \theta \right] + \left[ 2y + 1 \right] \left[ r \cos \theta \right]
\]

\[
= -2r^2 \cos \theta \sin \theta + 2r^2 \sin \theta \cos \theta + r \cos \theta
\]

\[
= r \cos \theta
\]

6. Let \( f(x, y) = x^2 + 2y^2 \).

a) Find \( \nabla f \)

\[
\nabla f = \left< 2x, 4y \right>
\]

b) Consider the level curve \( f(x, y) = 3 \). Find the positive \( x \) value that lies on the curve when \( y = -1 \).

\[
3 = x^2 + 2 \cdot (-1) = x^2 + 2
\]

\[
x = \pm 1 \quad \rightarrow \quad x = 1
\]

c) Evaluate \( \nabla f \) at that point.

\[
\nabla f (1, -1) = \left< 2, -4 \right>
\]

d) Sketch the level curve and \( \nabla f \) at that point.
7. A 2d biofilm growing on the bottom of a pipe has profile \( y = h(x) \).

a) Use the level curve formulation to find a general expression for the upward pointing normal, as shown in the figure.

\[
\bar{n} = \nabla F = \left\langle -h'(x), 1 \right\rangle
\]

b) If \( y = h(x) = 1 + \sin 2x \), write the normal vector at \( x = 0.1 \) using a calculator approximation for each component. Use radian mode and report 3 digits in your answer. Sketch the point and the normal vector on the plot.

\[
h' = 2 \cos 2x \quad h'(.1) = 1.96
\]

\[
\bar{n} = \left\langle -1.96, 1 \right\rangle
\]

8. Find the critical points of \( f(x,y) = x^2y + 3xy - \frac{3y^2}{2} + 2y + 1 \).

\[
f_x = 2xy + 3y = 0 = y (2x + 3)
\]

\[
y = 0 \quad \text{or} \quad x = -\frac{3}{2}
\]

\[
f_y = x^2 + 3x - 3y + 2 = 0
\]

\[
\begin{aligned}
&\text{if} \quad y = 0, \quad \text{then} \quad 0 = x^2 + 3x + 2 = (x+1)(x+2) \\
&\text{CP} \quad (-1,0) \quad (-2,0)
\end{aligned}
\]

\[
\begin{aligned}
&\text{If} \quad x = -\frac{3}{2}, \quad \text{then} \quad 0 = \frac{9}{4} - \frac{9}{2} - 3y + 2 \\
&= \frac{9}{4} + \frac{9}{4} - \frac{18}{4} - 3y \\
&= -\frac{1}{4} - 3y \\
&3y = -\frac{1}{4} \\
&y = -\frac{1}{12}
\end{aligned}
\]

\[
\text{CP} \quad \left(-\frac{3}{2}, -\frac{1}{12}\right)
\]
9. Uncle Max and Aunt Min are driving their Nissan Maxima to the Minimum Security Prison in the beautiful coastal village Critical Point to visit their wayward nephew Newton 'Vector' Lagrange. But they get lost and end up driving in circles – well, one circle, namely \( x^2 + y^2 = 10 \). The air temperature in the area is given by \( T(x,y) = 3x + 9y + 50 \). Use Lagrange Multipliers to find the warmest and coolest temperatures that Uncle Max and Aunt Min experience along their fruitless path.

\[
F(x, y, \lambda) = 3x + 9y + 50 - \lambda(x^2 + y^2 - 10)
\]

\[
F_x = 3 - 2\lambda x = 0 \quad \Rightarrow \quad 3 = 2\lambda x \quad \lambda = \frac{3}{2x}
\]

\[
F_y = 9 - 2\lambda y = 0 \quad \Rightarrow \quad 9 = 2\lambda y \quad \lambda = \frac{9}{2y}
\]

\[
\frac{3}{2x} = \frac{9}{2y} \quad \Rightarrow \quad 6y = 18x \quad \Rightarrow \quad y = 3x
\]

\[
x^2 + y^2 = 10
\]

\[
x^2 + 9x^2 = 10
\]

\[
x = \pm 1 \quad y = \pm 3
\]

\[
T(1,3) = 3 + 27 + 50 = 80
\]

\[
T(-1,-3) = -3 - 27 + 50 = 20
\]