- **Analytic functions**

  A function $f$ that has a power series representation

  \[ f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n, \]

  with a radius of convergence $\rho > 0$, is said to be **analytic** at $x_0$.

- **The Ratio Test**

  (i) If $\lim_{n \to \infty} \left| \frac{A_{n+1}}{A_n} \right| = L < 1$, then the series $\sum_{n=0}^{\infty} A_n$ converges absolutely. \(^1\)

  (ii) If $\lim_{n \to \infty} \left| \frac{A_{n+1}}{A_n} \right| = L > 1$ (including $L = \infty$), the series $\sum_{n=0}^{\infty} A_n$ diverges.

  (iii) If $\lim_{n \to \infty} \left| \frac{A_{n+1}}{A_n} \right| = 1$, no conclusion.

- **Ordinary and singular points**

  Consider the differential equation

  \[ P(x)y''(x) + Q(x)y'(x) + R(x)y(x) = 0, \]

  and divide through by $P(x)$. If $P(x_0) \neq 0$ and both $Q(x)/P(x)$ and $R(x)/P(x)$ are analytic at $x_0$, then $x_0$ is said to be an **ordinary point** of the differential equation. Otherwise $x_0$ is said to be a **singular point**.

- **Regular and irregular singular points**

  If $x_0$ is a singular point of the differential equation

  \[ P(x)y''(x) + Q(x)y'(x) + R(x)y(x) = 0, \]

  and $(x - x_0)Q(x)/P(x)$ and $(x - x_0)^2R(x)/P(x)$ are analytic at $x_0$, then $x_0$ is said to be a **regular singular point** of the differential equation.

  If $x_0$ is a singular point, but $(x - x_0)Q(x)/P(x)$ and $(x - x_0)^2R(x)/P(x)$ are not analytic at $x_0$, then $x_0$ is said to be an **irregular singular point**.

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\(^1\) Absolute convergence means $\sum_{n=0}^{\infty} |A_n|$ converges (and consequently $\sum_{n=0}^{\infty} A_n$ converges).
Differentiating a convergent power series

If \( \sum_{n=0}^{\infty} a_n (x - x_0)^n \) converges to \( f(x) \) for \( |x - x_0| < \rho, \rho > 0 \), then \( f \) is continuous and has derivatives of all orders for \( |x - x_0| < \rho \). Further, \( f', f'', \ldots \) can be computed by differentiating the series termwise, and each of the series converges absolutely for \( |x - x_0| < \rho \):

\[
f'(x) = \sum_{n=0}^{\infty} n a_n (x - x_0)^{n-1}, \quad f''(x) = \sum_{n=0}^{\infty} n(n-1) a_n (x - x_0)^{n-2}, \ldots
\]

Adding convergent power series

If \( \sum_{n=0}^{\infty} a_n (x - x_0)^n \) and \( \sum_{n=0}^{\infty} b_n (x - x_0)^n \) converge to \( f(x) \) and \( g(x) \), respectively, for \( |x - x_0| < \rho, \rho > 0 \), then the series can be added and subtracted termwise for \( x \) in the interval of convergence \( |x - x_0| < \rho \):

\[
f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n) (x - x_0)^n.
\]

That is, the series can be added just like polynomials. Similarly, the series can be multiplied and divided like polynomials.

EQUATING CONVERGENT POWER SERIES

If \( \sum_{n=0}^{\infty} a_n (x - x_0)^n \) and \( \sum_{n=0}^{\infty} b_n (x - x_0)^n \) converge for \( |x - x_0| < \rho, \rho > 0 \), and if \( \sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{n=0}^{\infty} b_n (x - x_0)^n \) for each \( x \) in the interval of convergence \( |x - x_0| < \rho \), then we can equate like terms: \( a_n = b_n \) for \( n = 0, 1, 2, \ldots \).

In particular, if \( \sum_{n=0}^{\infty} a_n (x - x_0)^n = 0 \) for each \( x \), then \( a_n = 0 \) for \( n = 0, 1, 2, \ldots \).

Rearrangements

If \( \sum a_n \) is an absolutely convergent series with sum \( s \), then any rearrangement of terms of \( \sum a_n \) has the same sum \( s \).

If \( \sum a_n \) is a conditionally convergent (convergent but not absolutely convergent) series with sum \( s \) and if \( r \) is ANY real number, then there is a rearrangement of the terms of \( \sum a_n \) that has the sum \( r \).