Article: Miscellaneous Definitions

This article is the repository of all definitions that don’t seem to fit elsewhere.

- Table of Contents

Some Pointers. Should you get lost, press the ‘Home’ key to return to this menu. Click on Main Menu to return to the main menu, or on tutorials to return to the main tutorial menu.

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Miscellaneous Definitions

Table of Contents

1. Various Number Systems
   • The Natural Numbers
   • The Integers
   • The Rational Numbers
   • The Real Numbers

2. Intervals
   • Various Intervals
   • Partitioning an Interval

3. Functions
   • Bounded Functions
1. Various Number Systems

In this section we give a brief discussion of the various number systems.

• The Natural Numbers

**Definition 1.1.** The set of *natural numbers*, denoted by \( \mathbb{N} \), is defined to be

\[ \mathbb{N} = \{ 1, 2, 3, 4, \ldots \} \]

The natural numbers are also known as the set of all *positive integers*.

• The Integers

**Definition 1.2.** The set of all *integers*, denoted by \( \mathbb{Z} \), is defined to be

\[ \mathbb{Z} = \{ \ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots \} \]

Thus, one might think of the integers, \( \mathbb{Z} \), as consisting of three distinct parts: The positive integers, \( \mathbb{N} \), the set of all natural numbers;
Section 1: Various Number Systems

the *negative integers*, which could be though of as $-\mathbb{N}$ (the negation of every number in $\mathbb{N}$); and 0. Symbolically,

$$\mathbb{Z} = -\mathbb{N} \cup \{0\} \cup \mathbb{N}.$$  

That’s a nice formula.

- **The Rational Numbers**

**Definition 1.3.** A number, $r$, is a *rational number* if it is the ratio of two integers; i.e. a rational number $r$ has the form

$$r = \frac{p}{q}, \quad p, q \in \mathbb{Z}, \quad q \neq 0.$$  

The set of all rational numbers is denoted $\mathbb{Q}$:

$$\mathbb{Q} = \{ r \mid r \text{ is a rational number} \}.$$  

- **The Real Numbers**
Definition 1.4. An irrational number is a number that cannot be expressed as the ratio of two integers. The set of all real numbers, denoted \( \mathbb{R} \), is defined by

\[
\mathbb{R} = \{ x \mid x \text{ is either a rational or irrational number} \}.
\]

In interval notation, \( \mathbb{R} \) is written as

\[
\mathbb{R} = (-\infty, +\infty).
\]

Within the context of Analytic Geometry, \( \mathbb{R} \) is also known as the \( x \)-axis.
2. Intervals

• Various Intervals

Definition 2.1. An interval $I$ is called an open interval if $I$ does not contain its endpoints.

Examples. There are two types: Intervals of finite length, and intervals of infinite length.

Intervals of Finite Length: Let $a < b$ be real numbers. Then

$$I = (a, b) = \{ x \in \mathbb{R} \mid a < x < b \}$$

is an open interval.

Intervals of Infinite Length: Let $a \in \mathbb{R}$. Then each of the following are open intervals.

$$( -\infty, +\infty ) = \{ x \in \mathbb{R} \mid -\infty < x < +\infty \} = \mathbb{R}$$

$$( -\infty, a ) = \{ x \in \mathbb{R} \mid -\infty < x < a \}$$

$$( a, +\infty ) = \{ x \in \mathbb{R} \mid a < x < +\infty \}$$
Another important type of interval is the closed interval.

**Definition 2.2.** An interval $I$ is called an closed interval if the endpoints of $I$ belong to the interval.

**Examples.** A general example is, for $a \leq b$,

$$I = [a, b] = \{ x \in \mathbb{R} \mid a \leq x \leq b \}$$

A particular example is $[0, 1]$.

**Symmetrical Intervals:** An interval $I$ of number is said to be symmetrical about the origin provided

$$x \in I \implies -x \in I.$$ 

Examples of symmetrical intervals are $(-1, 1)$, $[-3, 3]$, $(-\infty, \infty)$. The intervals are not symmetrical about the origin: $(-2, 3)$, $[1, 2]$, $[0, \infty)$. 
Section 2: Intervals

- **Partitioning an Interval**

**Definition 2.3.** Let \([a, b]\) be a closed interval. A *partition*, \(P\), of \([a, b]\) is any finite subset of \([a, b]\) that contains the numbers \(a\) and \(b\). Or, more symbolically, a finite set

\[
P = \{ x_0, x_1, x_2, \ldots, x_n \}
\]

is a partition of \([a, b]\) provided \(P \subseteq [a, b] \) and \(a, b \in [a, b]\). (Here, \(n \in \mathbb{N}\).)

**Definition Notes:** The labeling used in (1) is the standard way of symbolically writing the elements of a partition.

- The elements of a partition are called *partition points* or *nodes*.
- When we write the elements of a partition it is customary to have them labeled such that

\[
x_0 < x_1 < x_2 < \cdots < x_n.
\]

- With the convention established in the previous point, and the fact that \(a, b \in [a, b]\), it follows that \(x_0 = a\) and \(x_n = b\). Thus,

\[
a = x_0 < x_1 < x_2 < \cdots < x_n = b
\]
A visualization of a partition can be seen from the next diagram.

\[
\begin{align*}
&x_0 \quad x_1 \quad x_2 \quad x_3 \quad \cdots \quad x_{i-1} \quad x_i \quad \cdots \quad \cdots \quad x_n \\
&\text{Partitioning Scheme}
\end{align*}
\]

You can see from the chart above how the nodes partition, or subdivide the interval into pieces.

- If \( P \) is a partition as defined in (1), then the \( P \) also subdivides the interval into subintervals. These subintervals, for example, are used as a basis for the construction of the Definite Integral. The nodes of the partition \( P \) are used as endpoints of these subintervals. Below is a listing of the subinterval as well as the usual scheme for numbering them.

First Sub-interval: \( I_1 = [x_0, x_1] \).

Second Sub-interval: \( I_2 = [x_1, x_2] \).

Third Sub-interval: \( I_3 = [x_2, x_3] \).

Fourth Sub-interval: \( I_4 = [x_3, x_4] \).

\[\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \chalendary}

\]
Section 2: Intervals

The $i^{th}$ Sub-interval: $I_i = [x_{i-1}, x_i]$.

The $n^{th}$ Sub-interval: $I_n = [x_{n-1}, x_n]$.

From the above listing of the interval, it is clear the the partition, as given in equation (1), subdivides the interval into $n$ subintervals. This is the significance of the natural number $n$ in (1). You'll note that it takes $n + 1$ nodes to subdivide the interval $[a, b]$ into $n$ parts.

Finally, we note that the index variable, $i$, is used to manipulate the various elements of a partition: For $i = 1, 2, 3, \ldots, n$, the $i^{th}$ node is $x_i$ and the $i^{th}$ subinterval is $I_i = [x_{i-1}, x_i]$. The length of the $i^{th}$ subinterval, $I_i = [x_{i-1}, x_i]$, is typically denoted by the symbol $\Delta x_i$. The calculated length of the $i^{th}$ subinterval is given by the formula

$$\Delta x_i = x_i - x_{i-1},$$

which is nothing more than the value of the upper endpoint, $x_i$, of the $i^{th}$ subinterval minus the value of the lower endpoint, $x_{i-1}$ of this interval.
Example. Here is a simple example to illustrate. Let the interval $[a, b]$ be the $[0,1]$. The set

$$P = \{ 0, \frac{1}{3}, \frac{1}{2}, \frac{3}{4}, 1 \}$$

(2)
is a partition of the interval $[0, 1]$ since $P \subseteq [0, 1]$ and $0, 1 \in [0, 1]$. A natural question to ask is, “Where is all the elaborate label system?” The labeling system is there, you just have to use it.

Let’s index the nodes:

- $x_0 = 0$. ($x_0$ is always the left-hand endpoint.)
- $x_1 = \frac{1}{3}$.
- $x_2 = \frac{1}{2}$.
- $x_3 = \frac{3}{4}$.
- $x_4 = 1$. (The last node is always the right-hand endpoint.)

We can see now that $n = 4$. (The natural number $n$ is the index number of the right-hand endpoint; or more simply, $n$ is one-less the number of nodes in the partition—we have 5 nodes, so $n$ must be 4.)
Section 2: Intervals

Thus, the partition $P$ in equation (2) subdivides the interval $[0, 1]$ into $n = 4$ subintervals.

- **First Sub-interval**: $I_1 = [x_0, x_1] = [0, \frac{1}{3}]$.
- **Second Sub-interval**: $I_2 = [x_1, x_2] = [\frac{1}{3}, \frac{2}{3}]$.
- **Third Sub-interval**: $I_3 = [x_2, x_3] = [\frac{2}{3}, \frac{3}{4}]$.
- **Fourth Sub-interval**: $I_4 = [x_3, x_4] = [\frac{3}{4}, 1]$.

Lastly, the length of the $3^{\text{rd}}$ subinterval is obtained by taking the general formula

$$\Delta x_i = x_i - x_{i-1}$$

and putting $x = 3$,

$$\Delta x_3 = x_3 - x_2 = \frac{3}{4} - \frac{1}{2} = \frac{1}{4}.$$

Of course, in this simple example, we could have computed the length of the $3^{\text{rd}}$ subinterval by taking this interval $I_3 = [\frac{2}{3}, \frac{3}{4}]$, as computed above, and calculated its length $\Delta x_3 = \frac{3}{4} - \frac{1}{2} = \frac{1}{4}$. The first method is useful in abstract discussions, the latter is used for specific examples.
3. Functions

- **Bounded Functions**

**Definition 3.1.** Let $y = f(x)$ be a real-valued function having domain $\text{Dom}(f) \subseteq \mathbb{R}$. Let $A \subseteq \text{Dom}(f)$. We say that the function $f$ is *bounded* over the set $A$, if there is some number $M > 0$ such that

$$|f(x)| \leq M \quad \text{for all } x \in A. \quad (1)$$

In this case, we say that $M$ is a *bound* for $f$ over $A$ and that $f$ is bounded by $M$ over the set $A$.

*Definition Notes:* Algebraically, the absolute inequality in (1) is equivalent to

$$-M \leq f(x) \leq M \quad \text{for all } x \in A.$$

In terms of geometry, if we were to draw the graph of $f$ over the set $A$, and draw the horizontal lines $y = -M$ and $y = M$, then the graph of $f$ over the set $A$ does not go below the horizontal line $y = -M$ and does not go above the horizontal line $y = M$. 
Section 3: Functions

- Or, said more simply, a function $f$ is bounded over the set $A$ if the graph of $f$ lies between two horizontal lines.
- A function that is not bounded over a set $A$ is said to be unbounded over that set.
- It is convenient to create two related notions: bounded below and bounded above. A function $f$ is bounded below over $A$ if there exists a number $m$ such that $f(x) \geq m$, for all $x \in A$. A function $f$ is bounded above over $A$ if there exists a number $M$ such that $f(x) \leq M$, for all $x \in A$.
- The definition of boundedness can be rewritten: $f$ is bounded over the set $A$, if $f$ is both bounded above and bounded below over the set $A$.

Examples of Boundedness: The function $f(x) = x^2$ is bounded over the interval $[0, 1]$. Indeed, $|f(x)| \leq 1$ for all $x \in [0,1]$. We say, in this case, that $f$ is bounded above by 1, since $f(x) \leq 1$ for all $x \in [0,1]$, and bounded below be 0, since $f(x) \geq 0$ for all $x \in [0,1]$. The same function $f(x) = x^2$ is unbounded over the interval $(0, +\infty)$; however,
Section 3: Functions

over that interval, \( f \) is bounded below by 0 — it just doesn’t have an upper bound over that interval.