Instructions. The final exam will be on Monday, May 9, 2005, 8:00–9:55 am, in Ayer Hall, Room 112. This is a two-part final exam: The first part is basically test #4 (§§5.1–5.5; §§6.1–6.2) for which this document is a review; the second part is a review of the semester’s material (§§1.1–4.10), use the previous reviews and tests as well as your text and notes as resource documents.

The following is a selection of problems from the material to be covered on the first part of the final exam. These problems do not represent the entirely of the types of problems that may appear on the test. Solve these problems, ideally, without reference to your text.

1. State the Fundamental Theorem of Calculus, Part I.

Solution: If \( f \) is continuous on \([a, b]\), then the function \( g \) defined by

\[
g(x) = \int_a^x f(t) \, dt \quad a \leq x \leq b
\]

is continuous on \([a, b]\) and differentiable on \((a, b)\) and \( g'(x) = f(x) \).

2. Differentiate each of the following function.

(a) \( g(x) = \int_{-1}^x \sin(t^2) \, dt \)

Solution: From the Fundamental Theorem of Calculus, Part I, we have \( g'(x) = \sin(x^2) \)

(b) \( g(x) = \int_x^3 \sqrt{t^2 + 1} \, dt \)

Solution: We have, by the Fundamental Theorem of Calculus, Part I,

\[
g(x) = \int_x^3 \sqrt{t^2 + 1} \, dt = - \int_3^x \sqrt{t^2 + 1} \, dt
\]

\[
g'(x) = -\frac{1}{\sqrt{x^2 + 1}}
\]

(c) \( g(x) = \int_0^{3x^2} \tan(4t) \, dt \)

Solution: We use the Fundamental Theorem, in combination with the Chain Rule:

\[
g'(x) = \tan(4(3x^2)) \frac{d}{dx}(3x^2) = \tan(12x^2)(6x) = \frac{6x \tan(12x^2)}{2x}
\]

(d) \( g(x) = \int_{\sqrt{x}}^2 \sqrt{t^2 + 1} \, dt \)

Solution: As before, we first reverse the limits of integration then apply the Fundamental Theorem of Calculus, Part I:

\[
g(x) = \int_{\sqrt{x}}^2 \sqrt{t^2 + 1} \, dt = - \int_2^{\sqrt{x}} \sqrt{t^2 + 1} \, dt
\]

\[
g'(x) = -\frac{d}{dx}\left(\sqrt{x^2 + 1}\right) = -\sqrt{x + 1} \frac{1}{2\sqrt{x}}
\]

Thus,

\[
g'(x) = -\frac{\sqrt{x + 1}}{2\sqrt{x}} = -\frac{1}{2} \sqrt{\frac{x + 1}{x}}
\]

3. Elementary Integration. Evaluate each of the following definite and indefinite integrals.

(a) \( \int_1^2 6x^2 \, dx \)

Solution: \( \int_1^2 6x^2 \, dx = 2x^3 \bigg|_1^2 = 2(2^3 - 1^3) = 2(8 - 1) = 14 \)
(b) \[ \int_{-\pi/2}^{\pi/2} \cos(x) \, dx \]

**Solution:**
\[ \int_{-\pi/2}^{\pi/2} \cos(x) \, dx = \sin(x) \bigg|_{-\pi/2}^{\pi/2} = \sin(\pi/2) - \sin(-\pi/2) = 1 - (-1) = 2 \]

(c) \[ \int 7t^3 - 4 \sin(t) + 3 \csc^2(t) \, dt \]

**Solution:**
\[ \int 7t^3 - 4 \sin(t) + 3 \csc^2(t) \, dt = \frac{7}{4}t^4 + 4 \cos(t) - 3 \cot(t) + C \]

(d) \[ \int_{-1}^{2} |x - 1| \, dx \]

**Solution:** First, analyze the definition of \(|x - 1|\),
\[ |x - 1| = \begin{cases} 
  x - 1 & x - 1 \geq 0 \\
  -(x - 1) & x - 1 < 0 
\end{cases} \text{ or } |x - 1| = \begin{cases} 
  x - 1 & x \geq 1 \\
  1 - x & x < 1 
\end{cases} \]

We now break the interval of integration up as follows:
\[ \int_{-1}^{2} |x - 1| \, dx = \int_{-1}^{1} |x - 1| \, dx + \int_{1}^{2} |x - 1| \, dx = \int_{-1}^{1} (1 - x) \, dx + \int_{1}^{2} x - 1 \, dx \]
\[ = \left. \frac{1}{2} x^2 - x \right|_{-1}^{1} + \left. -\frac{1}{2} x^2 + x + \frac{1}{2} \right|_{1}^{2} \]
\[ = \frac{1}{2} + \frac{3}{2} + 0 - \left( -\frac{1}{2} \right) = 2 + \frac{1}{2} = 2 \frac{1}{2} = \frac{5}{2} \]

**Alternate Solution:** The above solution is time consuming. The region under consideration is just the union of two triangles. It is easy to compute the area of each triangle and add.

4. **Integration.** Evaluate each of the following definite and indefinite integrals.

(a) \[ \int \cos(3x) \, dx \]

**Solution:** We use the formula \( \int \cos(u) \, du = \sin(u) + C \). Let \( u = 3x \), \( du = 3 \, dx \), thus,
\[ \int \cos(3x) \, dx = \frac{1}{3} \int \cos(3x) \, 3x \, dx = \frac{1}{3} \sin(3x) + C \]

Alternatively, we can make the substitution,
\[ \int \cos(3x) \, dx = \frac{1}{3} \int \cos(3x) \, 3x \, dx = \frac{1}{3} \int \cos(u) \, du = \frac{1}{3} \sin(u) + C = \frac{1}{3} \sin(3x) + C \]

(b) \[ \int x^2(4x^3 + 1)^{3/2} \, dx \]

**Solution:** Here we use the Power Rule, \( \int u^n \, du = \frac{u^{n+1}}{n+1} + C \), let \( u = 4x^3 + 1 \), and so \( du = 12x^2 \, dx \). Thus,
\[ \int x^2(4x^3 + 1)^{3/2} \, dx = \frac{1}{12} \int (4x^3 + 1)^{3/2} \, 12x^2 \, dx = \frac{1}{12} \frac{2}{5} (4x^3 + 1)^{5/2} + C = \frac{1}{30} (4x^3 + 1)^{5/2} + C \]
(c) \[ \int \frac{4}{(3z+1)^3} \, dz \]

Solution: We apply the Power Rule, let \( u = 3z + 1 \) and so \( du = 3 \, dz \)

\[
\int \frac{4}{(3z+1)^3} \, dz = 4 \int (3z+1)^{-3} \, dz = \frac{4}{3} \int (3z+1)^{-3} \, dz \\
= \frac{4}{3} \left( \frac{1}{-2} \right) (3z+1)^{-2} + C = \frac{-2}{3} (3z+1)^{-2} + C
\]

(d) \[ \int_0^3 \sqrt{x+1} \, dx \]

Solution: Apply the Power Rule, with \( u = x + 1, \, du = dx \). We make a substitution, being sure to change the limits of integration.

\[
\int_0^3 \sqrt{x+1} \, dx = \int_0^4 u^{1/2} \, du = \frac{2}{3} u^{3/2} \bigg|_1^4 = \frac{2}{3} (4^{3/2} - 1^{3/2}) = \frac{2}{3} (8 - 1) = \frac{14}{3}
\]

(e) \[ \int \sec^2(5\theta) \, d\theta \]

Solution: Simple formula integral, \( \int \sec^2(u) \, du = \tan(u) + C \), where \( u = 5\theta \) and \( du = 5 \, d\theta \). Thus,

\[
\int \sec^2(5\theta) \, d\theta = \frac{1}{5} \int \sec^2(5\theta) \, 5 \, d\theta = \frac{1}{5} \tan(5\theta) + C
\]

5. Find the area of the region above the x-axis, under the graph of \( f(x) = x^2 + 1 \), between \( x = 1 \) and \( x = 2 \).

Solution: We have,

\[
A = \int_1^2 x^2 + 1 \, dx = \left[ \frac{x^3}{3} + x \right]_1^2 = \left( \frac{8}{3} + 2 \right) - \left( \frac{1}{3} + 1 \right) = \frac{10}{3}
\]

6. Find the area between the two curves \( f(x) = x^2 \) and \( g(x) = 8 - x^2 \).

Solution: The first is a parabola that opens up, and the second is one that opens down; they enclose an area. First, find where the two curves intersect. Put \( f(x) = g(x) \) and solve for \( x \):

\[
x^2 = 8 - x^2 \implies 2x^2 = 8 \implies x^2 = 4 \implies x^2 = \pm 2
\]

Thus,

\[
A = \int_{-2}^2 (8 - x^2) - x^2 \, dx = \int_{-2}^2 8 - 2x^2 \, dx = 2 \int_{-2}^2 4 - x^2 \, dx \\
= 2 \left[ 4x - \frac{x^3}{3} \right]_{-2} = 2 \left[ \left( 8 - \frac{8}{3} \right) - \left( -8 - \frac{-8}{3} \right) \right] \\
= 2 \left[ 16 - \frac{16}{3} \right] = 32 \left( 1 - \frac{1}{3} \right) = 32 \left( \frac{2}{3} \right) \\
= \frac{64}{3}
\]
7. Consider the region bounded by the two intersecting curves $y = x$ and $x = y^3$.
   (a) Set up the integral using $x$ as the variable of integration.
      Solution: First, represent the boundary curves as functions of $x$: $y = x$ and $y = \sqrt[3]{x}$. The two curves intersect at $(0, 0)$ and $(1, 1)$. Now set up with respect to $x$, we integrate the “upper” minus the “lower”:
      \[
      A = \int_0^1 \sqrt[3]{x} - x \, dx 
      \]
   (b) Set up the integral using $y$ as the variable of integration.
      Solution: We need to write the boundary curves as functions of $y$: $x = y$ and $x = y^3$. We integrate the “right” minus the “left”:
      \[
      A = \int_0^1 y - y^3 \, dy 
      \]

8. Consider the region bounded by the $x$-axis, the graph $y = 2x^2$, and the lines $x = 0$ and $x = 1$.
   (a) Rotate the region around the $x$ axis, find the volume of the generated solid.
      Solution: The cross-sectional areas are circles. If we slice the solid at $x$, the cross-sectional area is $A(x) = \pi(2x^2)^2 = 4\pi x^4$. Thus, we integrate the cross-sectional area function:
      \[
      V = \int_0^1 4\pi x^4 \, dx = \left. \frac{4\pi}{5} x^5 \right|_0^1 = \frac{4\pi}{5} 
      \]
   (b) Rotate the region around the $y$ axis, set up the volume integral of the generated solid.
      Solution: For any $y$, $0 \leq y \leq 2$, the cross-sectional area is a washer. The area of this washer is given by
      \[
      A(y) = \pi(1)^2 - \pi\left(\sqrt{\frac{y}{2}}\right)^2 = \pi \left(1 - \frac{y}{2}\right) 
      \]
      \[
      V = \int_0^2 \pi \left(1 - \frac{y}{2}\right) \, dy 
      \]

9. A solid $S$ has a semi-circular base bounded by the $x$ axis and the graph of $y = \sqrt{1 - x^2}$. Each cross section perpendicular to the $x$ axis is a square. Find the volume of the solid.
   Solution: If we cut the solid at $x$, $-1 \leq x \leq 1$, we obtain a square, the length of one side is given by $y = \sqrt{1 - x^2}$. Thus,
   \[
   V = \int_{-1}^1 (\sqrt{1 - x^2})^2 \, dx = \int_{-1}^1 1 - x^2 \, dx = \left. \left(1 - \frac{x^3}{3}\right) \right|_{-1}^1 = \left(1 - \frac{1}{3}\right) - \left(-1 - \frac{-1}{3}\right) = \frac{4}{3} 
   \]