Solutions (24 points) Below, please find a set of solutions to this assignment.

§6.3, pages 226–228.

(2 pts) 1. Problem 4. Prove that $\sum a_k$ A.C. implies $\sum a_k^2$ converges.

Proof: Since $\sum a_k$ is A.C., therefore, $\lim_{k \to \infty} |a_k| = 0$, by Theorem 6.2. There exists an $N$ such that $n \geq N$ implies $|a_k| < 1$. Now for $n \geq N$, $a_k^2 \leq |a_k|$. Since $\sum |a_k|$ converges, by the Comparison Test, we conclude $\sum a_k^2$ converges as well.

Random Thoughts: Define the following spaces of sequences:

$\ell_1 = \{ \{a_k\} | \sum |a_k| \text{ converges} \}$ and $\ell_2 = \{ \{a_k\} | \sum a_k^2 \text{ converges} \}$

These are well-known vector spaces of summable and square-summable sequences. This exercise simply proves that $\ell_2 \subseteq \ell_1$.

The space $\ell_2$ can be made into a Hilbert Space, with an appropriate definition for inner-product. (Problem 15, page 234 is related to the definition of inner product for $\ell_2$.)

(4 pts) 2. Problem 5. Let $\sum a_k$ be convergent.

(a) (2 pts) Prove that if $\sum a_k$ is A.C. and $\{b_k\}$ is bounded, then $\sum a_k b_k$ converges absolutely.

Proof: There exists an $B$ such that $|b_k| \leq B$, $\forall k \in \mathbb{N}$. Then

$|a_k b_k| \leq B |a_k| \quad \forall k \in \mathbb{N}$

We conclude from the Comparison Test that $\sum |a_k b_k|$ converges, which means $\sum a_k b_k$ converges absolutely.

(b) (2 pts) Suppose $\sum a_k$ is NAC\(^1\), show there exists a bounded sequence $\{b_k\}$ such that $\sum a_k b_k$ diverges.

Proof: Define a function $\text{sgn} : \mathbb{R} \to \mathbb{R}$ by

$$\text{sgn}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases} \quad \text{or, equivalently} \quad \text{sgn}(x) = \begin{cases} \frac{x}{|x|} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Note that for any $x \in \mathbb{R}$, $x \text{sgn}(x) = |x|$. Now define $b_k = \text{sgn}(a_k)$. Note that $\{b_k\}$ is a bounded sequence and that

$$\sum_{k=1}^{\infty} a_k b_k = \sum_{k=1}^{\infty} a_k \text{sgn}(a_k) = \sum_{k=1}^{\infty} |a_k|$$

diverges.

\(^1\) nonabsolutely convergent
3. Problem 12. Simple enough, just apply the various tests: (a) **Root** or **Ratio Test**; (b) **Ratio Test**; (c) **Ratio Test**; (d) **Ratio Test**; (d) **Root Test**; (f) **Root Test**. □

4. Problem 21. Suppose $\sum k a_k$ converges, give an example where $\sum a_k^2$ diverges.

**Example:** Consider the series,

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$$

By the **Alternating Series Test**, the series converges. Now consider

$$\sum_{k=1}^{\infty} \left( \frac{-1}{\sqrt{k}} \right)^2 = \sum_{k=1}^{\infty} \frac{1}{k}$$

which diverges, as it is a *p*-series, with $p = 1$. □

5. Problem 26. Let $\sum k a_k$ be nonabsolutely convergent, and let $\{p_k\}$ and $\{q_k\}$ be the nonnegative and nonpositive terms\(^2\) of $\{a_k\}$ respectively. Prove $\sum k p_k$ and $\sum k q_k$ diverge.

**Proof:** Let us set up the partial sum notation for each of these series:

$$s_n = \sum_{k=1}^{n} a_k \quad s'_n = \sum_{k=1}^{n} p_k \quad s''_n = \sum_{k=1}^{n} q_k$$

After noting that $a_k = p_k + q_k$, $\forall k \in \mathbb{N}$, we have, $s_n = s'_n + s''_n$, for all $n \in \mathbb{N}$.

We prove the contrapositive, suppose that either $\sum k p_k$ or $\sum k q_k$ converge. WLOG, assume $\sum k p_k$ converges, this means that the sequence of partial sums $\{s'_n\}$ is convergent. But this then implies $\{s''_n\}$ converges too, since

$$s''_n = s_n - s'_n$$

and both sequences on the right are convergent.

We conclude that if one of the two series converge, they both converge. Since $\sum k p_k$ and $\sum k q_k$ converge, we see, after observing $|a_k| = p_k - q_k$, for all $k \in \mathbb{N}$, that

$$\sum_{k=1}^{\infty} |a_k| = \sum_{k=1}^{\infty} (p_k - q_k) = \sum_{k=1}^{\infty} p_k - \sum_{k=1}^{\infty} q_k$$

also converges, which means that $\sum k a_k$ is not nonabsolutely convergent. The contrapositive is proven. □

\(^2\) $p_k = \frac{1}{2}(a_k + |a_k|)$ and $q_k = \frac{1}{2}(a_k - |a_k|)$