Solutions (21 points) Below, please find a set of solutions to this assignment.


(3pts) 1. Problem 10. Suppose \( f \in C([a, b]) \) s.t. \( \int_a^b f = 0 \). (Hint: For (a), perhaps use the Fundamental Theorem and the Mean Value Theorem; for (b), note that \( \int_a^x f \leq \int_a^b f \).)
   (a) Prove \( \exists c \in [a, b] \) s.t. \( f(c) = 0 \).
   \[\text{Proof: Define } F(x) = \int_a^x f, \text{by Theorem 5.15(a) (the Fundamental Theorem of Calculus), } F \text{ is differentiable on } [a, b], \text{ since } f \text{ is continuous on } [a, b]. \text{ Note that } F(a) = F(b) = 0 \text{ so by the Mean Value Theorem, } \exists c \in (a, b) \text{ s.t. } F'(c) = 0. \text{ But again by Theorem 5.15(a), } F'(c) = f(c), \text{ hence, } f(c) = 0. \]
   (b) If \( f \geq 0 \), then \( f = 0 \).
   \[\text{Proof: As in part (a), define } F(x) = \int_a^x f, \text{ now, for any } x \in [a, b],\]
   \[0 \leq f \implies \int_a^x 0 \leq \int_a^x f \implies 0 \leq F(x)\]
   By Theorem 5.5(c). Also, for any \( x \in [a, b] \), we have,
   \[0 \leq F(x) = \int_a^x f \leq \int_a^b f = 0.\]
   We have shown \( F(x) = 0, \forall x \in [a, b] \). By Theorem 5.15(a) and the fact that \( F \) is the zero function, \( f(x) = F'(x) = 0, \forall x \in [a, b] \). \[\square\]

(2pts) 2. Problem 11. Suppose \( f, g \in C([a, b]) \) s.t. \( \int_a^b f = \int_a^b g \). Prove \( \exists c \in [a, b] \) s.t. \( f(c) = g(x) \) (Hint: Apply Problem 10.)
   \[\text{Proof: Let } h = f - g, \text{ then } h \in C([a, b]). \text{ By Theorem 5.5(b),}\]
   \[\int_a^b h = \int_a^b (f - g) = \int_a^b f - \int_a^b g = 0\]
   The result now follows from Problem 10(a). \[\square\]

(3pts) 3. Problem 15. Find the derivative of \( F(x) = \int_0^{x^2} t^2 \sin(t^2) \, dt \)
   \[\text{Solution: This is an application of the Fundamental Theorem, and the Chair Rule. Define}\]
   \[G(x) = \int_0^{x^2} t^2 \sin(t^2) \, dt \text{ and } h(x) = x^2\]
Then \( F(x) = G(h(x)) \), so by the “Rule of Chains”, \( F'(x) = G'(h(x))h'(x) \). Now by the Fundamental Theorem, \( G'(x) = x^2 \sin(x^2) \), and we know \( h'(x) = 2x \). Thus,

\[
F'(x) = G'(h(x))h'(x) = G'(x^2))(2x) = (x^4 \sin(x^4))(2x)
\]

\[
F'(x) = 2x^5 \sin(x^4)
\]

(3pts) 4. Problem 18. Let \( f \in C([a,b]) \) and \( c \in (a,b) \). Define \( F(x) = \int_x^c f \). Prove \( F'(x) = f(x) \), for all \( x \in [a,b] \). (Be sure to prove that \( F \) is differentiable at \( c \) and that \( F'(c) = f(c) \).)

*Proof:* Note that by **Corollary 5.16**,

\[
F(x) = \int_x^c f = \int_x^a f + \int_a^c f = \int_x^a f - \int_a^c f
\]

After noting that second term in the right-hand expression is a fixed constant, so we are freely use **Theorem 5.15(a)**

\[
F'(x) = \frac{d}{dx} \int_x^a f - \frac{d}{dx} \int_a^c f = f(x) - 0 = f(x)
\]

Thus, \( F'(x) = f(x), \forall x \in [a,b] \). □

(3pts) 5. Problem 25. Define \( f(x) = x^2 \sin(1/x^2), x \neq 0 \), and \( f(0) = 0 \). Show \( f' \) is not necessarily integrable.

*Demonstration:* We have already seen that \( f \) is continuous and differentiable, even at \( x = 0 \), indeed,

\[
\left| \frac{1}{x} (f(x) - f(0)) \right| = |x \sin(1/x^2)| \leq |x|
\]

This implies \( f'(0) = 0 \). The complete derivative of \( f \) is

\[
f'(x) = \begin{cases} 
2x \sin(1/x^2) - (2/x) \cos(1/x^2) & x \neq 0 \\
0 & x = 0
\end{cases}
\]

However, \( f' \) is not differentiable over the interval \([0,1]\) since, obviously\(^1\), \( f' \) is unbounded over this interval.

(4pts) 6. Let \( f(x) = \begin{cases} 
|x| & -2 \leq x \leq 1 \\
x^2 + 1 & 1 < x \leq 2
\end{cases} \), and define \( F(x) = \int_{-2}^x f(t) \, dt \). Compute \( F(x) \), for \(-2 \leq x \leq 2 \).

*Solution:* The absolute value function is itself a piecewise defined function, so \( f \) can be rewritten as follows:

\[
f(x) = \begin{cases} 
-x & -2 \leq x \leq 0 \\
x & 0 \leq x \leq 1 \\
x^2 + 1 & 1 < x \leq 2
\end{cases}
\]

\(^1\)As demonstrated in class, you did take notes, didn’t you?
From this point, it is simple, but tedious to compute the derivative. Indeed, ...

For $-2 \leq x \leq 0$, \[ F(x) = \int_{-2}^{x} -x \, dx = \frac{1}{2} x^2 \bigg|_{-2}^{x} = \frac{1}{2} (4 - x^2), \]

For $0 \leq x \leq 1$, \[ F(x) = \int_{-2}^{x} f = \int_{-2}^{0} f + \int_{0}^{x} f \\
= F(0) + \int_{0}^{x} x \\
= 2 + \frac{1}{2} x^2, \]

For $1 \leq x \leq 2$, \[ F(x) = \int_{-2}^{x} f = \int_{-2}^{1} f + \int_{1}^{x} f \\
= F(1) + \int_{1}^{x} (x^2 + 1) \\
= \frac{5}{2} + (\frac{1}{3} x^3 + x) \bigg|_{1}^{x} = \frac{5}{2} + (\frac{1}{3} x^3 + x) - \left( \frac{1}{3} + 1 \right) \\
= \frac{1}{3} x^3 + x + \frac{7}{6}, \]

To summarize, \[ f(x) = \begin{cases} 
-x & -2 \leq x \leq 0 \\
x & 0 \leq x \leq 1 \\
x^2 + 1 & 1 < x \leq 2 
\end{cases} \quad F(x) = \begin{cases} 
2 - \frac{1}{2} x^2 & -2 \leq x \leq 0 \\
2 + \frac{1}{2} x^2 & 0 \leq x \leq 1 \\
\frac{1}{3} x^3 + x + \frac{7}{6} & 1 \leq x \leq 2 
\end{cases} \]

The graphs of the two functions are shown below. The graph of $F$ is shown as a solid curve, and the graph of $f$ is shown by a dashed curve.

Notice that the function $F$ is continuous—as the theory predicts—and there is a slight “kink” in the graph of $F$ at $x = 1$, this is a point of non-differentiability that is “caused” by the discontinuity of $f$ at $x = 1$. \[\square\]