Solutions (17 points) Below, please find a set of solutions to this assignment.

Before beginning the solutions, I thought I’d prove a fact that I have used numerous times in lecture, and is convenient to use in this assignment. In the future, this result can be used freely.

Proposition. Let \( f \) be bounded \([a,b]\), then \( \sup_{x,y\in[a,b]} |f(x) - f(y)| = \omega(f,[a,b]). \)

Proof: For simplicity, let \( M = \sup f(x) \) and \( m = \inf f(y) \) and recall \( \omega(f,[a,b]) = M - m \).

Define \( S = \{ |f(x) - f(y)| : x, y \in [a,b] \} \)

We want to prove \( \sup S = M - m \). To this end, let \( x, y \in [a,b] \) be arbitrary, observe that \( m - M \leq f(x) - f(y) \leq M - m \)

this implies \( |f(x) - f(y)| \leq M - m \quad \forall x, y \in [a,b] \) (2)

Now let \( \epsilon > 0 \). There are numbers \( x, y \in [a,b] \) such that \( f(x) > M - \epsilon/2 \) and \( f(y) < m + \epsilon/2 \)

By the above inequalities we have,

\[ |f(x) - f(y)| \geq f(x) - f(y) > (M - \epsilon/2) - (m + \epsilon/2) = M - m - \epsilon \] (3)

We have now shown in (2) that \( M - m \) is an upper bound to the set \( S \), as defined in (1), and that given any \( \epsilon > 0 \), there are elements \( x, y \in [a,b] \) such that (3) holds, that is, no number less than \( M - m \) is an upper bound of \( S \). This proves \( \sup S = M - m \), as desired. \( \square \)


(4pts) 1. Problem 19(a), (c). Undergraduates do part (a), graduates do part (c).

(a) Prove \( \omega(|f|,[a,b]) \leq \omega(f,[a,b]). \)

(c) Suppose \( 0 < m \leq f \). Prove \( \omega(1/f,[a,b]) \leq \omega(f,[a,b])/m^2. \)

§5.2, page 176–177.

(3pts) 2. Problem 9. Define \( g(x) = 1 + \sqrt{x}\sin(1/x^8) \), for \( x \neq 0 \) and \( g(0) = 1 \). Prove \( g \) is continuous, hence Riemann integrable on \([0,1]\).

(3pts) 3. Problem 11(b). Suppose \( f \) and \( g \) are monotone, prove \( f \circ g \in \mathcal{R}([c,d]). \)

(4pts) 4. Problem 13. (Use Problem 19(a), page 170.)

Prove that if \( f \in \mathcal{R}([a,b]) \), then \( |f| \in \mathcal{R}([a,b]) \) and \( \left| \int_a^b f \right| \leq \int_a^b |f|. \)

(3pts) 5. Problem 15. (Use Problem 19(c), page 170.)

Suppose \( g(x) \geq k > 0 \) and that \( f, g \in \mathcal{R}([a,b]) \), prove \( f/g \in \mathcal{R}([a,b]). \)
Solutions to Assignment #3

1. (a) From the Reverse Triangle Inequality, we have,
   \[ |f(x)| - |f(y)| \leq |f(x) - f(y)| \leq \omega(f, [a, b]) \quad \forall x, y \in [a, b] \]
   Or,
   \[ |f(x)| - |f(y)| \leq \omega(f, [a, b]) \quad \forall x, y \in [a, b] \]

   Now, by the Proposition above, we have
   \[ \omega(|f|, [a, b]) = \sup_{x,y \in [a,b]} |f(x)| - |f(y)| \leq \omega(f, [a, b]) \]
   and this proves the result.

1. (c) We have, for any \( x, y \in [a, b] \),
   \[ |f(x)| - |f(y)| \leq \frac{|f(x) - f(y)|}{f(x)f(y)} \leq \frac{|f(x) - f(y)|}{m^2} \leq \frac{\omega(f, [a, b])}{m^2} \]
   the last inequality is, once again, from the Proposition above. The inequality
   \[ \left| \frac{1}{f(x)} - \frac{1}{f(y)} \right| \leq \frac{\omega(f, [a, b])}{m^2} \]
   shows that \( \frac{\omega(f, [a, b])}{m^2} \) is an upper bound to the numbers on the left side of the above inequality.
   By the Proposition,
   \[ \omega(1/f, [a, b]) = \sup_{x,y \in [a,b]} \left| \frac{1}{f(x)} - \frac{1}{f(y)} \right| \leq \frac{\omega(f, [a, b])}{m^2} \]
   and this completes the proof.

2. Note that for \( x \neq 0 \),
   \[ |g(x) - g(0)| = |1 + \sqrt{x} \sin(1/x^8) - 1| = |\sqrt{x} \sin(1/x^8)| \leq \sqrt{x} \]
   Since \( \lim_{x \to 0^+} \sqrt{x} = 0 \), by the Squeeze Theorem it follows
   \[ \lim_{x \to 0^+} |g(x) - g(0)| = 0 \]
   this is enough to prove \( g \in C([0, 1]) \), and by Theorem 5.11, \( g \in R([0, 1]) \)

3. It is easy to prove that the composition of monotone functions is monotone, hence, \( f \circ g \in R([c, d]) \) by Theorem 5.12.

4. To show \( |f| \in R([a, b]) \), we use Theorem 5.10 and Exercise 19(a). Let \( \epsilon > 0 \), since \( f \in R([a, b]) \),
   there is a partition \( P \) such that
   \[ \sum_{i=1}^{n} \omega(f, [x_{i-1}, x_i])(x_i-x_{i-1}) < \epsilon \]
   From Exercise 19(a), we have,
   \[ \sum_{i=1}^{n} \omega(|f|, [x_{i-1}, x_i])(x_i-x_{i-1}) \leq \sum_{i=1}^{n} \omega(f, [x_{i-1}, x_i])(x_i-x_{i-1}) < \epsilon \]
   This proves \( |f| \in R([a, b]) \).
   Finally, note that \(-|f| \leq f \leq |f|\), pointwise, so by Theorem 5.5(c),
   \[ \int_a^b -|f| \leq \int_a^b f \leq \int_a^b |f| \]
and by Theorem 5.5(a),
\[-\int_a^b |f| \leq \int_a^b f \leq \int_a^b |f|\]
from which it follows,
\[\left| \int_a^b f \right| \leq \int_a^b |f|\]
as required.

\[\square\]

5. Because the product of two Riemann integrable functions is Riemann integrable, it suffices to prove \(1/g \in \mathcal{R}([a, b])\).

Let \(\epsilon > 0\), since \(g \in \mathcal{R}([a, b])\), there is a partition \(P\) such that
\[\sum_{i=1}^n \omega(g, [x_{i-1}, x_i])(x_i - x_{i-1}) < k^2 \epsilon\]  \hspace{1cm} (4)

For notational purposes, define \(k_i = \inf\{ g(x) : x \in [x_{i-1}, x_i]\}\), and note that \(k \leq k_i\), for each \(i\).

For the same partition \(P\) as in (4), by Exercise 19(c) applied to each subinterval of the partition \(P\), we have,
\[\sum_{i=1}^n \omega(1/g, [x_{i-1}, x_i])(x_i - x_{i-1}) \leq \sum_{i=1}^n \frac{1}{k_i^2} \omega(g, [x_{i-1}, x_i])(x_i - x_{i-1})\]
\[\leq \frac{1}{k^2} \sum_{i=1}^n \omega(g, [x_{i-1}, x_i])(x_i - x_{i-1})\]
\[< \frac{1}{k^2} \cdot k^2 \epsilon\]
\[= \epsilon\]

C’est ça.  \hspace{1cm} \square