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Problem 2. Prove that \( \frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k \), \( |x| < 1 \).

Solution: The sum of a geometric series (Proposition 9.6) is

\[
\frac{1}{1-r} = \sum_{k=0}^{\infty} r^k, \quad |r| < 1 \tag{1}
\]

Let \( x \in \mathbb{R} \) such that \( |x| < 1 \), and put \( r = -x \) in equation (1) above, note that \( |r| = |-x| = |x| < 1 \) so, by (1), we have

\[
\frac{1}{1-(-x)} = \sum_{k=0}^{\infty} (-x)^k, \quad |-x| < 1
\]

or, simplifying, we have

\[
\frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k, \quad |x| < 1 \tag{2}
\]

which is the advertised equation. \( \square \)

Problem 3. Prove that

\[
\frac{1}{(1+x)^2} = \sum_{k=0}^{\infty} (-1)^k k x^{k-1}, \quad |x| < 1 \tag{*}
\]

Solution: This will be a little hard as the stated equation is not true. (Notice that if you evaluate both sides of (\( * \)) at \( x = 0 \) you get \( 1 = -1 \).) Let’s see what is true. From the previous problem we know,

\[
\frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k, \quad |x| < 1 \tag{1}
\]

We use Theorem 9.24 to deduce, for \( |x| < 1 \),

\[
\frac{d}{dx} \frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k
\]

\[
-\frac{1}{(1+x)^2} = \sum_{k=0}^{\infty} \frac{d}{dx} (-1)^k x^k
\]

\[
= \sum_{k=0}^{\infty} (-1)^k k x^{k-1}
\]
The last series above is the one that appears on the right-hand side of (*). These calculations show it is not equal to the left-hand side of (*). We’ll continue to simplify. Multiplying both sides by $-1$, and distributing through the sum, we get

$$
\frac{1}{(1 + x)^2} = \sum_{k=0}^{\infty} (-1)^{k+1}kx^{k-1}
$$

(2)

This is almost, but not quite equation (*), now let’s reindex

$$
\frac{1}{(1 + x)^2} = \sum_{k=1}^{\infty} (-1)^{k+1}kx^{k-1}
$$

remove $k = 0$ term

Let $j = k - 1$, or $k = j + 1$

$$
\frac{1}{(1 + x)^2} = \sum_{j=0}^{\infty} (-1)^{j+2}(j+1)x^{j}
$$

Let $j = k - 1$, or $k = j + 1$

$$
= \sum_{k=0}^{\infty} (-1)^{k+2}(k+1)x^{k}
$$

Replace $j$ with $k$

$$
= \sum_{k=0}^{\infty} (-1)^{k}(k+1)x^{k}
$$

Since $(-1)^{k+2} = (-1)^{k}$

(3)

Again, almost (*), but not quit!

To summarize, we have shown, from (2) and (3),

$$
\frac{1}{(1 + x)^2} = \sum_{k=1}^{\infty} (-1)^{k-1}kx^{k-1} = \sum_{k=0}^{\infty} (-1)^{k}(k+1)x^{k}
$$

(4)

I think we’re done. □

**Problem 5.** Prove that $x = \sum_{k=0}^{\infty} \left( 1 - \frac{1}{x} \right)^k$, if $|1 - x| < |x|$.

**Solution:** This series is derived from the geometric series, equation (1) of Problem 1. Put $r = (1 - 1/x)$. For the geometric series to converge, we must have $|r| < 1$, thus, $x$ must be such that $|1 - 1/x| < 1$, which after simple manipulation, yields $|x - 1| < |x|$. Thus, for any $x$ such that $|x - 1| < |x|$, we have

$$
\frac{1}{1 - (1 - \frac{1}{x})} = \sum_{k=0}^{\infty} \left( 1 - \frac{1}{x} \right)^k
$$

But a simple calculation shows that

$$
\frac{1}{1 - (1 - \frac{1}{x})} = x
$$

and the claim is proved. □

**Problem 6.** Define $f(x) = 1/(1 - x)^3$, if $|x| < 1$. Find a power series representation of $f$.

**Solution:** From the geometric series, we have,

$$
\frac{1}{1 - x} = \sum_{k=0}^{\infty} x^k \quad |x| < 1
$$

Now, differentiate both sides, using Theorem 9.24

$$
\frac{1}{(1-x)^2} = \sum_{k=1}^{\infty} kx^{k-1} \quad |x| < 1
$$
Differentiate again, using Theorem 9.24
\[
\frac{2}{(1-x)^3} = \sum_{k=2}^{\infty} k(k-1)x^{k-2} \quad |x| < 1
\]
Thus, ...
\[
\frac{1}{(1-x)^3} = \sum_{k=2}^{\infty} \frac{k(k-1)}{2} x^{k-2} \quad |x| < 1
\]
Reindex, let \( j = k - 2 \)
\[
\frac{1}{(1-x)^3} = \sum_{j=0}^{\infty} \frac{(j+1)(j+2)}{2} x^j \quad |x| < 1
\]
Replace \( j \) with \( k \)
\[
\frac{1}{(1-x)^3} = \sum_{k=0}^{\infty} \frac{(k+1)(k+2)}{2} x^k \quad |x| < 1
\]
Summary:
\[
f(x) = \frac{1}{(1-x)^3} = \sum_{k=0}^{\infty} \frac{(k+1)(k+2)}{2} x^k \quad |x| < 1
\]

**Problem 7.** Define \( f(x) = \sum_{k=0}^{\infty} c_k x^k, |x| < r \). Let \([a, b]\) be contained in the interval \((-r, r)\). Prove
\[
\int_a^b f(x) \, dx = \sum_{k=0}^{\infty} \frac{c_k}{k+1} [b^{k+1} - a^{k+1}].
\]

Solution: Seems simple enough. Let \( x_0 \) be a number such that \(-r < -x_0 < a < b < x_0 < r\). Then \( x_0 \) is in the domain of convergence of the power series.

Now choose \( R \) so that \(-r < -x_0 < -R < a < b < R < x_0 < r\). We do this manipulation to position ourselves to use Theorem 9.23, we conclude the series \( \sum_{k=0}^{\infty} c_k x^k \) converges uniformly on \([-R, R]\), hence uniformly on \([a, b]\). We are now in the position to call on Theorem 9.18—actually the in-class corollary, a version stated in terms of series—to conclude that “term-by-term” integration is permissible. Thus, ...
\[
\int_a^b f(x) \, dx = \int_a^b \sum_{k=0}^{\infty} c_k x^k \, dx = \sum_{k=0}^{\infty} \int_a^b c_k x^k \, dx
\]
\[
= \sum_{k=0}^{\infty} \frac{c_k}{k+1} x^{k+1} \bigg|_a^b = \sum_{k=0}^{\infty} \frac{c_k}{k+1} [b^{k+1} - a^{k+1}]
\]
C’est tout. □

**Problem 8.** Obtain a series expansion of for the integral \( \int_0^{1/2} \frac{1}{1+x^4} \, dx \).

Solution: Simple enough, again. Let \( |x| < 1 \). In the formula for geometric series, equation (1) of Problem 1, put \( r = -x^4 \), then \(|r| = |x^4| < 1\), so substitution is valid. We have,
\[
\frac{1}{1+x^4} = \sum_{k=0}^{\infty} (-x^4)^k = \sum_{k=0}^{\infty} (-1)^k x^{4k} \quad |x| < 1
\]
Now, from Problem 7, we have,
\[
\int_0^{1/2} \frac{1}{1 + x^4} \, dx = \int_0^{1/2} \sum_{k=0}^{\infty} (-1)^k x^{4k} \, dx = \sum_{k=0}^{\infty} (-1)^k \int_0^{1/2} x^{4k} \, dx
\]
\[
= \sum_{k=0}^{\infty} (-1)^k \frac{x^{4k+1}}{4k + 1}\bigg|_0^{1/2} = \sum_{k=0}^{\infty} (-1)^k \frac{(1/2)^{4k+1}}{4k + 1}
\]
\[
= \sum_{k=0}^{\infty} \frac{(-1)^k}{4k + 1} \frac{1}{2^{4k+1}}
\]
Thus,
\[
\int_0^{1/2} \frac{1}{1 + x^4} \, dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{4k + 1} \frac{1}{2^{4k+1}}
\]
and this completes the assignment. □