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**Problem 1 (b), (d).** Make derivative calculations of indefinite integrals.

(b) Compute \( \frac{d}{dx} \int_1^{e^{x}} \ln t \, dt \)

*Solution:* Apply Corollary 7.6

\[
\frac{d}{dx} \int_1^{e^{x}} \ln t \, dt = \ln(e^{x}) \frac{d}{dx} e^{x} = xe^{x}
\]

(d) Compute \( \frac{d}{dx} \int_1^{x} \cos(x + t) \, dt \)

*Solution:* The integrand is a function of \( x \) and \( t \), we must separate the integrand:

\[
\int_1^{x} \cos(x + t) \, dt = \int_1^{x} \cos(x) \cos(t) - \sin(x) \sin(t) \, dt
\]

\[
= \cos(x) \int_1^{x} \cos(t) \, dt - \sin(x) \int_1^{x} \sin(t) \, dt
\]

Differentiating . . .

\[
\frac{d}{dx} \left( \int_1^{x} \cos(x + t) \, dt \right) = \frac{d}{dx} \left( \cos(x) \int_1^{x} \cos(t) \, dt \right) - \frac{d}{dx} \left( \sin(x) \int_1^{x} \sin(t) \, dt \right)
\]

applying the product rule and second fund. thm. of Calc . . .

\[
= \cos^2(x) - \sin(x) \int_1^{x} \cos(t) \, dt - \sin^2(x) - \cos(x) \int_1^{x} \sin(t) \, dt
\]

\[
= \cos^2(x) - \sin(x)(\sin(x) - \sin(1)) - \sin^2(x) - \cos(x)(- \cos(x) + \cos(1))
\]

\[
= 2 \cos^2(x) - 2 \sin^2(x) + \sin(x) \sin(1) - \cos(x) \cos(1)
\]

\[
= 2 \cos(2x) - \cos(x + 1)
\]

*Check:* Let’s check our answer . . .

\[
\frac{d}{dx} \int_1^{x} \cos(x + t) \, dt = \frac{d}{dx} \left( \sin(x + t) \right)_1^{x} = \frac{d}{dx} (\sin(2x) - \sin(x + 1))
\]

\[
= 2 \cos(2x) - \cos(x + 1)
\]

Ahhh! Most satisfying. □

There is a well-known theorem, called the **Leibnitz Rule.** Under suitable conditions we have,

\[
\frac{d}{dx} \int_{\alpha(x)}^{\beta(x)} f(x, t) \, dt = \frac{d}{dx} (f(x, \beta(x)) - \alpha'(x) f(x, \alpha(x)) + \int_{\alpha(x)}^{\beta(x)} f_x(x, t) \, dt)
\]

Let’s check out this formula for this problem. \( f(x, t) = \cos(x + t), f_x(x, t) = - \sin(x + t) \). Thus,

\[
\frac{d}{dx} \left( \int_1^{x} \cos(x + t) \, dt \right) = (1) \cos(x + x) - (0) \cos(x + 0) + \int_1^{x} - \sin(x + t) \, dt
\]

\[
= \cos(2x) + \cos(x + t) \bigg|_1^{x} = \cos(2x) + \cos(x + x) - \cos(x + 1)
\]

\[
= 2 \cos(2x) - \cos(x + 1)
\]

That worked! □
Problem 2 (b). Define \( f(x) = \begin{cases} x^2 & 0 \leq x \leq 1 \\ x & 1 < x \leq 2 \end{cases} \), compute \( F(x) = \int_0^x f(t) \, dt \).

Solution: For \( 0 \leq x \leq 1 \) we have

\[
F(x) = \int_0^x f(t) \, dt = \int_0^x t^2 \, dt = \frac{1}{3}x^3
\]

For \( 1 < x \leq 2 \),

\[
F(x) = \int_0^x f(t) \, dt = \int_0^1 f(t) \, dt + \int_1^x f(t) \, dt
\]

\[= F(1) + \int_1^x t \, dt = \frac{1}{3} + \frac{1}{2}(x^2 - 1)\]

\[= \frac{1}{2}x^2 - \frac{1}{6}\]

Thus,

\[
F(x) = \begin{cases} \frac{1}{3}x^3 & 0 \leq x \leq 1 \\ \frac{1}{2}x^2 - \frac{1}{6} & 1 < x \leq 2 \end{cases}
\]

That completes the calculation. \( \square \)

Problem 3. Define \( H(x) = \int_{-x}^x [f(t) + f(-t)] \, dt \), compute \( H''(x) \).

Solution: We have,

\[
H'(x) = \frac{d}{dx} \int_{-x}^x [f(t) + f(-t)] \, dt
\]

\[= \frac{d}{dx} \int_0^x [f(t) + f(-t)] \, dt + \frac{d}{dx} \int_0^{-x} [f(t) + f(-t)] \, dt
\]

\[= -\frac{d}{dx} \int_0^x [f(t) + f(-t)] \, dt + \frac{d}{dx} \int_0^x [f(t) + f(-t)] \, dt
\]

\[= -[f(-x) + f(x)](-1) + [f(x) + f(-x)]\]

\[= 2[f(x) + f(-x)]\]

Differentiating again . . .

\[
H''(x) = 2[f'(x) - f'(-x)]
\]

Problem 4. Show that \( f(x) = f(0) + f'(0)x + \int_0^x (x-t) f''(t) \, dt \).

Solution: Define

\[
G(x) = f(x) - f(0) - f'(0)x - \int_0^x (x-t) f''(t) \, dt
\]

\[= f(x) - f(0) - f'(0)x - x \int_0^x f''(t) \, dt + \int_0^x t f''(t) \, dt
\]

We need to show that \( G(x) = 0, \forall x \in \mathbb{R} \). Note \( G(0) = 0 \). We now compute the derivative of \( G \).

\[
G'(x) = f'(x) - f'(0) - xf''(x) - \int_0^x f''(t) \, dt + xf''(x)
\]

\[= f'(x) - f'(0) - \int_0^x f''(t) \, dt = f'(x) - f'(0) - f'(0)\big|_0^x\]

\[= f'(x) - f'(0) - (f'(x) - f'(0)) = 0\]

The Identity Criterion then implies that \( G(x) \) is a constant function. Since \( G(0) = 0 \), we see that \( G(x) = 0 \). This proves the stated identity. \( \square \)
Problem 6. Define \( F(x) = \int_1^x \frac{1}{2\sqrt{t}-1} \, dt \). Prove that if \( c > 0 \) there is a unique solution to the equation \( F(x) = c, \ x > 1 \).

Solution: We have \( F'(x) = \frac{1}{2\sqrt{x}-1} > 0, \ x > 1 \). This shows \( F \) is strictly increasing. From this we see that any solution to the equation \( F(x) = c \) is unique. To show existence, observe that \( F(1) = 0 \) and \( \frac{1}{2\sqrt{x}-1} \geq \frac{1}{2\sqrt{x}} \). Hence, from the monotonicity property,

\[
F(x) = \int_1^x \frac{1}{2\sqrt{t}-1} \, dt \geq \int_1^x \frac{1}{2\sqrt{t}} \, dt = \sqrt{x} - 1
\]

Thus, \( F(x) \geq \sqrt{x} - 1, \) for all \( x \geq 1 \). Now, for any \( c > 0, \) obviously, there exists a \( x_0 > 1 \) such that \( \sqrt{x_0} - 1 \geq c \). This shows that \( F(0) = 0 < c \leq \sqrt{x_0} - 1 \leq F(x_0) \)

By Proposition 7.2, \( F \) is continuous, and by the Intermediate Value Theorem, there is some \( x, \) \( 1 < x \leq x_0 \) such that \( F(x) = c \). \( \square \)

Problem 9. Define \( p(x) = a_1 x + a_2 x^2 + \cdots + a_n x^n \) and suppose

\[
\frac{a_1}{2} + \frac{a_2}{3} + \cdots + \frac{a_n}{n+1} = 0.
\]

Prove that there is a number \( x \in (0, 1) \) such that \( p(x) = 0 \).

Proof: Let \( F(x) = \int_0^x p(t) \, dt = \frac{a_1}{2} x^2 + \frac{a_2}{3} x^3 + \cdots + \frac{a_n}{n+1} x^{n+1} \). By the Second Fundamental Theorem, we have \( F'(x) = p(x) \). Also note that \( F(0) = 0 \), and it is given from (1) that \( F(1) = 0 \). So, by Rolle’s Theorem, there exists \( x_0 \in (0, 1) \) such that \( F'(x_0) = 0 \), i.e., \( p(x_0) = 0 \). \( \square \)

Problem 10. Generalize the Mean Value Theorem for Integrals.

Proof: We use the same methods as in Problem 9. Since \( f \) is continuous, by the Second Fundamental Theorem, there is a function \( F: [a, b] \to \mathbb{R} \) such that \( F'(x) = f(x), \) for \( x \in (a, b) \). From the First Fundamental Theorem, we have

\[
\int_a^b f(x) \, dx = F(b) - F(a)
\]

Apply the Mean Value Theorem for Differentiation to the function \( F : [a, b] \to \mathbb{R} \), we have the existence of a number \( x_0 \in (a, b) \) such that

\[
F(b) - F(a) = F'(x_0)(b - a)
\]

Putting (1) and (2) together with the fact that \( F' = f \) on \( (a, b) \), we have

\[
\int_a^b f(x) \, dx = F(b) - F(a) = F'(x_0)(b - a) = f(x_0)(b - a)
\]

or, there exists a number \( x_0 \in (a, b) \) such that \( \frac{1}{b-a} \int_a^b f(x) \, dx = f(x_0) \). This completes the proof. \( \square \)