
**Problem 1.** Define \( f(x) = x, \ x \in [0, 1] \), compute upper and lower sums for a regular partition

**Solution:** For a regular partition \( P_n \), we have \( x_i = i/n, \ 0 \leq i \leq n \). Utilizing the fact that \( f \) is increasing we see that \( m_i = x_i - 1 = (i - 1)/n \) and \( M_i = x_i = i/n, \ 1 \leq i \leq n \). For a regular partition, \( \Delta x_i = x_i - x_{i-1} = 1/n \).

\[
L(f, P_n) = \sum_{i=1}^{n} m_i \Delta x_i = \frac{1}{n} \sum_{i=1}^{n} \frac{i-1}{n} = \frac{1}{n^2} \sum_{i=1}^{n} (i-1)
\]

\[
= \frac{1}{n^2} \frac{n(n-1)}{2} = \frac{n-1}{2n} \tag{1}
\]

and

\[
U(f, P_n) = \sum_{i=1}^{n} M_i \Delta x_i = \frac{1}{n} \sum_{i=1}^{n} \frac{i}{n} = \frac{1}{n^2} \sum_{i=1}^{n} i
\]

\[
= \frac{1}{n^2} \frac{n(n+1)}{2} = \frac{n+1}{2n} \tag{2}
\]

From (1) and (2), we see that

\[
U(f, P_n) - L(f, P_n) = \frac{1}{n}
\]

It follows that \( f \in \mathcal{R}([0, 1]) \) from the Integrability Criterion. □

**Problem 3.** Easy, choose \( n \) so large that \((b-a)/n < \delta\), then build a regular partition \( P_n \) based on \( n \) subdivisions.

**Problem 4.** If \( f(x) \geq 0 \) for all \( x \in [a, b] \), then \( m = \inf \{ f(x) \mid x \in [a, b] \} \geq 0 \) too. Take \( P = \{a, b\} \) as the partition, then

\[
\int_{a}^{b} f \geq L(f, P) = m(b-a) \geq 0
\]

and that completes the proof. □

**Problem 6.** Define \( f(x) = \begin{cases} x & 2 \leq x \leq 3 \\ 0 & 3 \leq x \leq 4 \end{cases} \). Prove \( f \in \mathcal{R}([2, 4]) \).

**Proof:** Let \( \epsilon > 0 \). Using the technique introduced into class, we argue as follows: Since \( f: [2, 3] \rightarrow \mathbb{R} \) is continuous, we have \( f \in \mathcal{R}([2, 3]) \), by the Integrability Criterion, there exists a partition \( P_1 \) of \( [2, 3] \) such that

\[
U(f, P_1) - L(f, P_1) < \epsilon/2 \tag{1}
\]

Now, \( f: [3, 4] \rightarrow \mathbb{R} \) is bounded and \( f: (2, 3) \rightarrow \mathbb{R} \) is continuous, so by Corollary 6.7, \( f \in \mathcal{R}([3, 4]) \). So, there exists a partition \( P_2 \) of \( [3, 4] \) such that

\[
U(f, P_2) - L(f, P_2) < \epsilon/2 \tag{2}
\]

Define \( P = P_1 \cup P_2 \), then from (1) and (2), we have

\[
U(f, P) - L(f, P) = (U(f, P_1) - L(f, P_1)) + (U(f, P_2) - L(f, P_2)) < \epsilon/2 + \epsilon/2 = \epsilon
\]

**Alternate Solution:** A couple of students made a more direct argument. Create a regular partition by subdividing \( [2, 4] \) into \( 2n \), an even number of subintervals. Then, a direct calculation of the upper and lower sums is possible, leading to the result. □
Problem 12. Define a function, \( f \) on \([0, 1]\), as follows: \( f(x) = x \) if \( x \) is rational, and \( f(x) = -x \) if \( x \) is irrational. Prove that \( f \notin \mathcal{R}([0, 1]) \).

Proof: Let \( P \) be any partition of \([0, 1]\), and let \( P^* = P \cup \{1/2\} \) be a refinement of \( P \). It is easy to see that

\[
U(f, P) - L(f, P) \geq U(f, P^*) - L(f, P^*) = \sum_{i=1}^{n} 2x_i^* \Delta x_i
\]

There \( x_i^* \) are the partition points of \( P^* \). Now

\[
U(f, P) - L(f, P) \geq \sum_{i=1}^{n} 2x_i^* \Delta x_i \geq \sum_{i:x_i^* \geq 1/2} 2x_i^* \Delta x_i
\]

\[
\geq \sum_{i:x_{i-1} \geq 1/2} 2 \cdot \frac{1}{2} \Delta x_i = \sum_{i:x_{i-1} \geq 1/2} \Delta x_i
\]

\[
\geq \frac{1}{2}
\]

Thus, \( U(f, P) - L(f, P) \geq 1/2 \), for all partitions \( P \); by the Integrability Criterion, \( f \) is not integrable. \( \square \)

Alternate Solution: Using some ideas from §6.3, we can also argue as follows: as before,

\[
U(f, P) - L(f, P) = \sum_{i=1}^{n} 2x_i \Delta x_i
\]

The right-hand side we recognize as the upper Riemann sum of the function \( g(x) = 2x \), thus,

\[
U(f, P) - L(f, P) = \sum_{i=1}^{n} 2x_i \Delta x_i = U(g, P)
\]

We know, from the First Fundamental Theorem—or by direct calculation of Upper/Lower Riemann-Darboux sums that

\[
\int_{0}^{1} 2x \, dx = 1
\]

Thus,

\[
U(f, P) - L(f, P) = \sum_{i=1}^{n} 2x_i \Delta x_i = U(g, P) \geq \int_{0}^{1} 2x \, dx = 1
\]

Thus, if fact, \( U(f, P) - L(f, P) \geq 1 \), a better inequality than the one obtained earlier. \( \square \)

Problem 15. Assume \( f \in \mathcal{R}([a, b]) \) and bounded away from zero, \( f(x) \geq m > 0 \) for all \( x \in [a, b] \). Prove \( 1/f \in \mathcal{R}([a, b]) \) by proving that for any partition \( P \),

\[
U(1/f, P) - L(1/f, P) \leq \frac{1}{m^2} [U(f, P) - L(f, P)]
\]

Proof: Assume the usual notation for a given partition \( P \). Note that for any \( i \), \( 1 \leq i \leq n \),

\[
m_i \leq f(x) \leq M_i \implies \frac{1}{M_i} \leq \frac{1}{f(x)} \leq \frac{1}{m_i}, \quad x \in [x_{i-1}, x_i]
\]

The latter set of inequalities implies

\[
\frac{1}{M_i} \leq m_i^* \leq M_i^* \leq \frac{1}{m_i}
\]

where \( M_i^* \) and \( m_i^* \) are the sup and inf, respectively, of the function \( 1/f \) over the \([x_{i-1}, x_i]\), and consequently, we have

\[
M_i^* - m_i^* \leq \frac{1}{m_i} - \frac{1}{M_i}
\]

\(^1\)We may have equality here, but equality is not needed, so I won’t go to the trouble of proving equality.
Finally note that \( m \leq m_i \leq M_i \) implies

\[
\frac{1}{M_i} \leq \frac{1}{m_i} \leq \frac{1}{m}, \quad \text{for } 1 \leq i \leq n
\]

(4)

Thus,

\[
U(1/f, P) - L(1/f, P) = \sum_{i=1}^{n} \left( M_i^* - m_i^* \right) \Delta x_i \leq \sum_{i=1}^{n} \left( \frac{1}{m_i} - \frac{1}{M_i} \right) \Delta x_i \quad \text{from (3)}
\]

\[
\leq \sum_{i=1}^{n} \frac{M_i - m_i}{M_i m_i} \Delta x_i \leq \sum_{i=1}^{n} \frac{M_i - m_i}{m^2} \Delta x_i \quad \text{from (4)}
\]

\[
\leq \frac{1}{m^2} \sum_{i=1}^{n} (M_i - m_i) \Delta x_i
\]

\[
= \frac{1}{m^2} \left[ U(f, P) - L(f, P) \right]
\]

This completes the proof of the inequality. To see that \( 1/f \) is integrable, let \( \epsilon > 0 \). Choose a partition \( P \) such that \( U(f, P) - L(f, P) < m^2 \epsilon \), then for that same partition, we have

\[
U(1/f, P) - L(1/f, P) \leq \frac{1}{m^2} \left[ U(f, P) - L(f, P) \right] < \frac{1}{m^2} m^2 \epsilon = \epsilon
\]

By the Integrability Criterion, \( 1/f \in \mathcal{R}([a, b]) \). □