Instructions Test #2 will be Tuesday, October 19, 2004. The test covers the following sections in the text: §§3.3–3.10. The following are a selection of problems from the material to be covered on the test. These problems do not represent the entirely of the types of problems you may appear on the test. The solutions are at the end of the document. Solve the problems first, then look at the solutions.

1. (Elementary) Calculate each of the following. Simplify as needed.

(a) \( \frac{d}{dx} (4x^3 - 5x^2 + 1) = \)
Solution:
\[
\frac{d}{dx} (4x^3 - 5x^2 + 1) = 12x^2 - 10x = 2x(6x - 5)
\]

(b) \( \frac{d}{dt} \left( \sqrt{2t} - 6t^{3/2} + \frac{4}{\sqrt{t}} \right) = \)
Solution:
\[
\frac{d}{dt} \left( \sqrt{2t} - 6t^{3/2} + \frac{4}{\sqrt{t}} \right) = \frac{d}{dt} \left( \sqrt{2t} - 6t^{3/2} + 4t^{-1/2} \right)
= \sqrt{\frac{2}{2}} t^{-1/2} - 9t^{1/2} - 2t^{-3/2}
= t^{-3/2} \left( \sqrt{\frac{2}{2}} t - 9t^2 - 2 \right)
= -t^{-3/2} (9t^2 - \sqrt{\frac{2}{2}} t + 2)
\]

(c) \( \frac{d}{d\theta} (4\sin(\theta) - 3\csc(\theta)) = \)
Solution:
\[
\frac{d}{d\theta} (4\sin(\theta) - 3\csc(\theta)) = 4\cos(\theta) + 3\csc(\theta) \cot(\theta)
\]

(d) \( \frac{d}{dx} \left( \frac{3x^3 - 6x^2}{\sqrt{x}} \right) = \)
Solution:
\[
\frac{d}{dx} \left( \frac{3x^3 - 6x^2}{\sqrt{x}} \right) = \frac{d}{dx} (3x^{5/2} - 6x^{3/2})
= \frac{15}{2} x^{3/2} - 9x^{1/2}
= \frac{1}{2} x^{1/2} (15x - 18)
= \frac{1}{2} \sqrt{x} (15x - 18)
\]

2. (Advanced) Differentiate each of the following. Simplify as needed.

(a) \( \frac{d}{dx} x \tan(x) = \)
Solution:
\[
\frac{d}{dx} x \tan(x) = x \sec^2(x) + \tan(x)
\]

(b) \( \frac{d}{dx} \frac{x^2}{\cos(x)} = \)
Solution:
\[
\frac{d}{dx} \frac{x^2}{\cos(x)} = \frac{\cos(x)(2x^2) - x^2(-\sin(x))}{\cos^2(x)}
= \frac{2x \cos(x) + x^2 \sin(x)}{\cos^2(x)}
\]

(c) \( \frac{d}{dx} (3x^3 + 1)^5 = \)
Solution:
\[
\frac{d}{dx} (3x^3 + 1)^5 = 5(3x^3 + 1)^4 (9x^2)
= 45x^2 (3x^3 + 1)^4
\]

(d) \( \frac{d}{dt} \sec(5t^3) = \)
Solution:
\[
\frac{d}{dt} \sec(5t^3) = 15t^2 \sec(5t^3) \tan(5t^3)
\]

(e) \( \frac{d}{dx} \tan^3(4\sqrt{x}) = \)
Solution:
\[
\frac{d}{dx} \tan^3(4\sqrt{x}) = 3\tan^2(4\sqrt{x}) \frac{d}{dx} (\tan(4\sqrt{x}))
= 3\tan^2(4\sqrt{x}) \sec^2(4\sqrt{x}) \frac{d}{dx} (4\sqrt{x})
= 3\tan^2(4\sqrt{x}) \sec^2(4\sqrt{x}) \frac{2}{\sqrt{x}}
\]
Thus,
\[
\frac{d}{dx} \tan^3(4\sqrt{x}) = \frac{3}{2\sqrt{x}} \tan^2(4\sqrt{x}) \sec^2(4\sqrt{x}) \frac{2}{\sqrt{x}}
\]

(f) \[
\frac{d}{dx} \frac{x^2}{x^2 + 1} = 2x \frac{x^2 - 1}{(x^2 + 1)^2}
\]

Solution: Apply the quotient rule:
\[
\frac{d}{dx} \frac{x^2}{x^2 + 1} = (x^2 + 1)(2x) - (x^2)(2x)
\]

3. Find the equation of the line tangent to the graphs of each of the following function at the indicated points.

(a) \(f(x) = 3x^2 - 4x\) at \(x = 2\).
Solution: We have \(f(2) = 4\) and \(f'(x) = 6x - 4\) and so \(f'(2) = 8\). Now, using the point-slope form of the equation of a line we have...
\[
y - 4 = 8(x - 2) \text{ or } y = 8x - 12
\]

(b) \(y = \sin(x)\) at \(x = \pi/3\).
Solution: Same exercise as before, but with different details. \(f'(\pi/3) = \sqrt{3}/2\), and \(f'(x) = \cos(x)\) so \(f'(\pi/3) = 1/2\). Thus,
\[
y - \frac{\sqrt{3}}{2} = \frac{1}{2} \left( x - \frac{\pi}{3} \right)
\]

4. A particle moves along the \(s\) axis. At time \(t\) its position is given by \(s = 4t^3 - 12t\). Find the velocity and acceleration at time \(t = 2\).
Solution: We have \(v = s' = 12t^2 - 12\) and \(a = v' = s'' = 24t\). Thus, at time \(t = 2\) we have
\[
v(2) = 36 \text{ and } a(2) = 48
\]

5. Use implicit differentiation to compute \(dy/dx\) for each of the following equations. You can use \(y'\) as an alternate notation to \(dy/dx\).

(a) \(x^2 \sin(y) = 1\)
Solution:
\[
x^2 \sin(y) = 1
\]
\[
\frac{d}{dx} x^2 \sin(y) = 0
\]
\[
x^2 \frac{d}{dx} \sin(y) + \frac{d}{dx} x^2 \sin(y) = 0
\]
\[
x^2 \cos(y) y' + 2x \sin(y) = 0
\]
\[
y' = -\frac{2x \sin(y)}{x^2 \cos(y)}
\]
\[
y' = -\frac{2 \tan(y)}{x}
\]

(b) Find the slope of the line tangent to the graph of the equation \(3y - x^2 y^3 = 1\) at the point, \((2, -1)\).
Solution:
\[
3y - x^2 y^3 = 1
\]
\[
\frac{d}{dx} 3y - x^2 y^3 = 10
\]
\[
3y' - 3x^2 y^2 y' - 2x y^3 = 0
\]
\[
(3 - 3x^2 y^2) y' = 2x y^3
\]
\[
y' = \frac{2x y^3}{3(1 - x^2 y^2)}
\]
Thus,
\[
y'(2, -1) = \frac{2(2)}{3(1 - 4)} = \frac{-4}{9}
\]

6. Find \(y''\) for the equation \(x^2 - y^2 = 1\) using implicit differentiation methods.
Solution: Differentiate both sides to obtain,
\[
2x - 2yy' = 0
\]
Solving for \(y'\) we get \[y' = \frac{x}{y}\]

7. If a snowball melts so that its surface area decreases at a rate of 1 cm\(^2\)/min, find the rate at which the diameter decreases when the diameter is 10 cm.
Solution: Let \(S\) denote the surface area of the snowball, and \(D\) the diameter. We are given that \(dS/dt = -1\). We went to find \(dD/dt\). We must interrelate \(S\) and \(D\) at the algebraic level.

The surface area of a sphere is \(S = 4\pi r^2\), where \(r\) is the radius. But \(r = \frac{D}{2}\). Thus,
\[
S = 4\pi \left(\frac{D}{2}\right)^2 = \pi D^2
\]
Differentiate with respect to time, \(t\), to obtain,
\[
\frac{dS}{dt} = 2\pi D \frac{dD}{dt} \quad \text{or} \quad \frac{dD}{dt} = \frac{dS/dt}{2\pi D}
\]
Plugging in our given info we get
\[
\frac{dD}{dt} \bigg|_{D=10} = \frac{1}{2\pi(10)} = \frac{1}{20\pi} \text{ cm/min}
\]

8. Suppose a 6-ft tall person is 12 ft away from an 18-ft tall lamppost. If the person is moving away from the lamppost at a rate of 2 ft/sec, at what rate is the length of the shadow changing?
Solution: Let \(x\) denote the distance the person is away from the lamppost, and let \(s\) be the length of the his/her shadow. If you draw the picture we have similar right triangles, so
\[
\frac{18}{x + s} = \frac{6}{s}
\]
or \(18s = 6(x + s)\), and so \(s = \frac{1}{2}x\). Differentiating with respect to \(t\) we get
\[
\frac{ds}{dt} = \frac{1}{2} \frac{dx}{dt}
\]
You can see by this equation that the rate at which the length of the shadow grows, is one-half the rate at which the person walks (moves away from the lamppost). Finally, since \(dx/dt = 2\) ft/sec, we have
\[
\frac{ds}{dt} = 1 \text{ ft/sec}
\]

9. Calculate the differential of each of the following,
(a) \( y = \sin(x^2) \)

Solution: \( dy = 2x \cos(x^2) \, dx \)

(b) \( A = \pi r^2 \)

Solution: \( dA = 2\pi r \, dr \)

10. The edge of a cube was found to be 40 cm with a possible error in measurement of 0.2 cm. Use differentials to estimate the maximum possible error in computing the volume of the cube. (Note: For uniformity of notation, let \( V \) denote the volume of the cube and let \( x \) denote the length of one of the sides of the cube.)

Solution: We have that \( V = x^3 \) and so \( dV = 3x^2 \, dx \). Now

\[ |dV| \leq 3(40)^2(0.2) = 960 \text{ (cm}^3) \]

The maximum error in the volume is 960 cm\(^3\)

11. Find the linearization, \( L(x) \), of each function, \( f \), at the specified point, \( a \).

(a) \( f(x) = \frac{1}{\sqrt{2 + x}} \) at \( a = 2 \).

Solution: We use the formula

\[ L(x) = f(a) + f'(a)(x - a) \]

for \( a = 2 \). We have \( f(2) = 1/2 \). Also,

\[
\begin{align*}
    f'(x) & = \frac{d}{dx}(2 + x)^{-1/2} \\
    & = -\frac{1}{2}(2 + x)^{-3/2} \\
    & = -\frac{1}{2(2 + x)^{3/2}}
\end{align*}
\]

Thus, \( f'(2) = -1/16 \). Now substituting into the formula we obtain

\[ L(x) = \frac{1}{2} - \frac{1}{16}(x - 2) \]

Or,

\[ L(x) = \frac{5}{8} - \frac{x}{16} \]

(b) \( f(x) = \sin(x) \) at \( a = \pi/4 \).

Solution: Some problem as the previous one, we just repeat the steps. \( f(\pi/4) = \sqrt{2}/2 \). Also, \( f'(x) = \cos(x) \) so \( f'(\pi/4) = \sqrt{2}/2 \), too. Thus,

\[ L(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \left( x - \frac{\pi}{4} \right) \]

Not too pretty. :-)

12. Compute the limits of each of the following, show all details.

(a) \( \lim_{x \to 0} \frac{\sin 4x}{\sin 5x} = \)

Solution:

\[
\begin{align*}
    \lim_{x \to 0} \frac{\sin 4x}{\sin 5x} & = \lim_{x \to 0} \frac{(\sin 4x)/x}{(\sin 5x)/x} \\
    & = \lim_{x \to 0} \frac{4(\sin 4x)/(4x)}{5(\sin 5x)/(5x)} \\
    & = \lim_{x \to 0} \frac{4}{5} \left( \frac{\sin 4x}{4x} \right) \left( \frac{\sin 5x}{5x} \right) \\
    & = \frac{4}{5}
\end{align*}
\]

(b) \( \lim_{x \to 0} \frac{\tan(6x)}{2x} = \)

Solution:

\[
\begin{align*}
    \lim_{x \to 0} \frac{\tan(6x)}{2x} & = \frac{6}{2} \lim_{x \to 0} \frac{\tan(6x)}{6x} \\
    & = 3 \lim_{x \to 0} \frac{\sin(6x)}{6x} \frac{1}{\cos(6x)} \\
    & = (3)(1)(1) = 3
\end{align*}
\]