Definitions

1. Give a precise definition of each of the terms below.

   (a) Let \( \{ x_n \} \) be a sequence of real numbers, define what it means for \( \{ x_n \} \) to be a Cauchy sequence.
   \( \text{Solution:} \forall \epsilon > 0, \exists N \in \mathbb{N}, \text{such that} \quad |x_m - x_n| < \epsilon, \forall m, n \geq N. \)

   (b) Let \( I \) be an open interval, \( c \in I \) and \( f : I \setminus \{ c \} \to \mathbb{R} \). Define what it means for \( f \) to have a limit \( L \) at \( c \).
   \( \text{Solution:} \forall \epsilon > 0, \exists \delta > 0, \text{such that} \quad (\forall x \in I)(0 < |x - c| < \delta \implies |f(x) - L| < \epsilon) \)

Statements of Theorems and Quick Responses

1. State the Bolzano-Weierstrass Theorem
   \( \text{Solution:} \) Any bounded sequence has a convergent subsequence. Furthermore, if the sequence lies in a bounded interval \([ a, b \] ), then there is a subsequence that converges to an element of the interval \([ a, b \] ).

2. True/False, and fill in the blank. No justification needed. Write ‘T’ or ‘F’ in the space provided.
   (a) \( \text{T} \) If \( \{ x_n \} \) is a Cauchy sequence, then \( \{ x_n \} \) is bounded.
   (b) \( \text{F} \) Let \( \{ x_n \} \) be a sequence, and suppose two subsequences of \( \{ x_n \} \) have the same limit, then \( \{ x_n \} \) converges.
   (c) \( \text{T} \) The limit superior of a bounded sequence is the largest subsequential limit point of that sequence.
   (d) \( \text{T} \) Let \( \{ x_n \} \) be a bounded sequence and \( L \in \mathbb{R} \). If every subsequence of \( \{ x_n \} \) converges to \( L \), then \( \{ x_n \} \) converges.
   (e) \( \text{F} \) Let \( f : [a, b] \to \mathbb{R} \) be a function and \( c \in [a, b] \). Suppose \( \lim_{x \to c} f(x) = L \) exists, then \( L = f(c) \).
   (f) \( \text{T} \) Let \( f \) be a function defined on \([ a, b \] \), and \( c \in [a, b] \). Suppose there is an \( \epsilon > 0 \) and a sequence \( \{ x_n \} \) converging to \( c \) such that \( |f(x_n) - f(c)| \geq \epsilon, \forall n \in \mathbb{N} \). Then \( f \) not continuous at \( c \).
   (g) Let \( f \) be a function that is discontinuous at \( c \in \mathbb{R} \). If \( f(c^-) = f(c^+) \), then we say that \( f \) has a \underline{removable} discontinuity at \( c \).

4. For each \( n \in \mathbb{N} \), defined \( a_n = 1 + (-1)^n + 2^{-n} \). Compute the limit inferior and limit superior of this sequence. Present your answers using correct notation. (No justification needed.)
   \( \text{Solution:} \)
   \( \lim \inf a_n = 0 \quad \text{and} \quad \lim \sup a_n = 2 \)

- Solve the rest of the test on separate sheets of paper. Be sure to label each of your problems clearly and put them in the proper order.
Proofs of Theorems

(16 pts) 5. Prove exactly one of the following two theorems. The proofs should be well written.

(a) If \( \{x_n\} \) is a bounded monotone sequence, then \( \{x_n\} \) converges.
   
   Solution: As done in class and the text.

(b) Let \( I \) be an open interval that contains the point \( c \) and let \( f : I \setminus \{c\} \rightarrow \mathbb{R} \) be a function. Suppose that for each sequence \( \{x_n\} \) in \( I \setminus \{c\} \) that converges to \( c \), the sequence \( \{f(x_n)\} \) converges to \( L \), prove that \( f \) has limit \( L \) at \( c \).
   
   Solution: As done in class and the text.

Problems

(15 pts ea.) 6. Solve exactly one of the following two problems.

(a) Let \( \{a_n\} \) be an unbounded sequence. Then there exists a subsequence \( \{a_{p_n}\} \) of \( \{a_n\} \) such that \( \{1/a_{p_n}\} \) converges to zero.
   
   Solution: We construct the subsequence by induction. There exists an natural number \( p_1 \) such that \( |a_{p_1}| > 1 \). Suppose \( p_1 < p_2 < \cdots < p_n \) have been chosen so that \( |a_{p_k}| > k \), for \( k = 1, 2, \ldots, n \). Since \( n + 1 \) is not an upper bound for the sequence \( \{a_n\} \), there exists \( p_{n+1} \in \mathbb{N} \) such that \( |a_{p_{n+1}}| > n + 1 \). We can choose \( p_{n+1} > p_n \), because if such an index did not exist, it would follow that the sequence \( \{a_n\} \) is bounded. (Verify?) This completes the induction and the construction.

   We have constructed a subsequence \( \{a_{p_n}\} \) of \( \{a_n\} \) with the property that \( |a_{p_n}| > n \), for all \( n \in \mathbb{N} \). It follows that \( |1/a_{p_n}| < 1/n \), or \( -1/n < 1/a_{p_n} < 1/n \). From the Squeeze Theorem, we deduce \( \{1/a_{p_n}\} \) converges to zero.

(b) Using the \((\epsilon, \delta)\) definition of limit,

   - if you are enrolled at the 400 level of this course, prove

   \[
   \lim_{x \to 2} \frac{2x}{3x + 1} = \frac{4}{7}
   \]

   Solution: Let \( \epsilon > 0 \). As usual, we seek an \( \delta < 1 \). We begin by investigating...

   \[
   \left| \frac{2x}{3x + 1} - \frac{4}{7} \right| = \left| \frac{2x - 4}{7} \frac{1}{3x + 1} \right| < \frac{1}{4} |x - 2| \tag{1}
   \]

   Here, we have used the fact that \( 4/7 < 1 \) and, since \( \delta < 1 \), \( 1 < x < 3 \) and so \( 4 < 3x + 1 < 10 \).

   It is clear that if we choose \( 0 < \delta < \min\{ 1, 4\epsilon \} \), then for \( 0 < |x - 2| < \delta \), we have, from (1),

   \[
   \left| \frac{2x}{3x + 1} - \frac{4}{7} \right| < \frac{1}{4} |x - 2| < \frac{1}{4} \delta < \frac{1}{4} (4\epsilon) = \epsilon
   \]

   - if you are enrolled at the 500 level of this course, prove

   \[
   \lim_{x \to 1} \frac{2x^2}{3x^2 + x + 1} = \frac{2}{5}
   \]

   Solution: Let \( \epsilon > 0 \). As usual, we seek an \( \delta < 1 \). We begin by investigating...

   \[
   \left| \frac{2x^2}{3x^2 + x + 1} - \frac{2}{5} \right| = \frac{2 |x - 1| \cdot |2x - 1|}{5 (3x^2 + x + 1)} \quad \text{algebra manipulation}
   \]

   \[
   = \frac{2}{5} |x - 1| \cdot 5 \quad \text{since} \; |2x - 1| < 2x + 1 < 5, \; 3x^2 + x + 1 \geq 1
   \]

   \[
   < 2 |x - 1| \tag{2}
   \]
Now, if we take $0 < \delta < \min\{1, \epsilon/2\}$, then for $0 < |x - 1| < \delta$, from (2), we have,

$$\left|\frac{2x^2}{3x^2 + x + 1} - \frac{2}{5}\right| < 2|x - 2| < 2(\epsilon/2) = \epsilon$$

(15 pts) 7. Solve exactly one of the following two problems.

(a) Let $\{a_n\}$ be a monotone sequence such that $\{a_n^2\}$ converges. Prove that $\{a_n\}$ converges too.

Solution: Since $\{a_n^2\}$ is convergent, it is bounded, hence $\exists M$ such that $a_n^2 \leq M$ for all $n \in \mathbb{N}$. But, this implies $|a_n| \leq \sqrt{M}$, for all $n \in \mathbb{N}$.

We have now shown that the monotone sequence $n \in \mathbb{N}$ is bounded, hence is convergent.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a function and let $c \in \mathbb{R}$. Assume $\lim_{x \to c} f(x) = L$. For any number $a \neq 0$, using $\epsilon$-$\delta$ definition of limit, the prove that $\lim_{x \to \frac{c}{a}} f(ax) = L$.

Solution: For $\epsilon > 0$, there is a $\delta_1 > 0$ such that

$$0 < |x - c| < \delta_1 \implies |f(x) - L| < \epsilon$$

Define $\delta = \delta_1/|a|$, then

$$0 < |x - c/a| < \delta \implies 0 < |ax - c|/|a| < \delta_1/|a| \implies 0 < |ax - c| < \delta_1 \implies |f(ax) - L| < \epsilon$$