Here is my take on these problems, of course, each can be solved in a variety of ways.

§3.4, pages 114–115.

**Problem 2.** Those enrolled at the 400 level of this course, do parts (b) and (c); for those enrolled at the 500 level, do parts (d) and (e).

(b) Prove \( f(x) = x^2 + 2x - 5 \) is u.c. on \([0, 3]\).

*Proof:* Let \( \epsilon > 0 \) be given, let \( \delta < \epsilon / 8 \), then for \( x, y \in [0, 3] \), with \( |x - y| < \delta \), we have,

\[
|f(x) - f(y)| = |(x^2 + 2x - 5) - (y^2 + 2y - 5)|
= |(x^2 - y^2) + 2(x - y)|
\leq (x + y)|x - y| + 2|x - y|
\leq 8|x - y| \quad x, y \in [0, 3] \implies x + y \leq 6
< 8 \frac{\epsilon}{8} = \epsilon \quad \Box
\]

(c) Prove \( h(x) = 4/x^2 \) is u.c. on \([1, 5]\).

*Proof:* Let \( \epsilon > 0 \) be given, let \( \delta < \epsilon / 40 \), then for \( x, y \in [1, 5] \), with \( |x - y| < \delta \), we have,

\[
|h(x) - h(y)| = |4/x^2 - 4/y^2|
= |4|(x^2 - y^2)/(x^2y^2)|
\leq 4(x + y)|x - y| \quad \text{since } x \geq 1 \text{ and } y \geq 1
\leq 4 \cdot 10|x - y| \quad x, y \in [1, 5] \implies x + y \leq 10
< 40 \frac{\epsilon}{40} = \epsilon \quad \Box
\]

(d) Prove \( F(x) = \sqrt{x} \) is u.c. on \([0, \infty)\).

*Proof:* Let \( \epsilon > 0 \). Since \( \sqrt{x} \) is an increasing function, we have

\[
0 \leq x < \epsilon^2 / 4 \implies \sqrt{x} < \epsilon / 2 \quad (1)
\]

Let \( 0 < \delta < \epsilon^2 / 4 \). Let \( x, y \in [0, \infty) \) such that \( |x - y| < \delta \).

**Case 1:** \( x, y < \delta \). Then from (1) we have

\[
|\sqrt{x} - \sqrt{y}| \leq \sqrt{x} + \sqrt{y} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad (2)
\]
Case 2: Either $x \geq \delta$ or $y \geq \delta$. We assume, WLOG, that $y \geq \delta$. Then,

$$|\sqrt{x} - \sqrt{y}| \leq \frac{|x - y|}{\sqrt{x} + \sqrt{y}} \leq \frac{\delta}{\sqrt{\delta}} = \sqrt{\delta} < \frac{\epsilon}{2} < \epsilon$$ (3)

In all cases, we have shown (see (2) and (3)) that $|\sqrt{x} - \sqrt{y}| < \epsilon$, whenever $|x - y| < \delta$. \qed

Solution Notes: Here is a nicer proof of the result, as presented by one of the students\(^1\) in the class. First prove the inequality\(^2\) $|\sqrt{x} - \sqrt{y}| \leq \sqrt{|x - y|}$, for $x, y \in [0, \infty)$. For $\epsilon > 0$, choose $\delta < \epsilon/2$. Then for $x, y \in [0, \infty)$, s.t. $|x - y| < \delta$,

$$|\sqrt{x} - \sqrt{y}| < \sqrt{|x - y|} < \sqrt{\delta} < \sqrt{\epsilon^2} = \epsilon. \quad \square$$

(d) Prove the function $G(x) = x^3$ is not u.c. on $[0, \infty)$.

Proof: Let $\epsilon = 1$. For each $\delta > 0$ we need to find $x, y \in [0, \infty)$ such that $|x - y| < \delta$, but $|x^3 - y^3| \geq 1$.

For each $n \in \mathbb{N}$, define $y_n = n$ and $x_n = y_n + 1/n$. Note that $|x_n - y_n| = 1/n$, and

$$|x_n^3 - y_n^3| = (x_n^2 + x_n y_n + y_n^2)(x_n - y_n) \geq 3y_n^2(1/n) \geq 3n^2(1/n) = 3n \geq 1$$

Thus, $|x_n - y_n| \to 0$, but $|x_n^3 - y_n^3| \geq 1$. This is enough to show $G$ is not uniformly continuous on $[0, \infty)$. \qed

Problem 4. For $\epsilon > 0$, choose $\delta = \epsilon/M$. \quad \square

Problem 7. (Uniform continuity is preserved by composition of functions.)

Proof: Let $\epsilon > 0$.

Now $f$ is u.c., and $\epsilon > 0$ implies there exists $\delta_1 > 0$ such that

$$(y_1, y_2 \in J) \land (|y_1 - y_2| < \delta_1) \implies |f(y_1) - f(y_2)| < \epsilon \quad (1)$$

$g$ is u.c., and $\delta_1 > 0$ implies there exists $\delta > 0$ such that

$$(x_1, x_2 \in I) \land (|x_1 - x_2| < \delta) \implies |g(x_1) - g(x_2)| < \delta_1 \quad (2)$$

Finally, let $x_1, x_2 \in I$ such that $|x_1 - x_2| < \delta$, then $|g(x_1) - g(x_2)| < \delta_1$, by (2). But by (1),

$$|f(g(x_1)) - f(g(x_2))| < \epsilon, \quad \text{since} \quad |g(x_1) - g(x_2)| < \delta_1$$

This proves the uniform continuity of $f \circ g$. \quad \square

\(^1\)Actually, there were multiplicities of students that had an identical argument! Recall: You need to be doing these problems on your own.

\(^2\)This solution also illustrates the power of developing just the right inequality.
Problem 8. (A uniformly continuous function defined on a bounded interval is bounded.)

Proof: Extend the definition of \( f \) to include the endpoints: Define
\[
g(x) = \begin{cases} 
  f(a^-) & x = a \\
  f(x) & x \in (a, b) \\
  f(b^+) & x = b 
\end{cases}
\]

The one-sided limits exist by Theorem 3.30.

It is easy to argue that \( g \) is continuous on \([a, b]\). But \( g \) continuous on \([a, b]\) implies \( g \) is bounded on \([a, b]\). Finally, \( g \) bounded on \([a, b]\) implies \( g \) is bounded on \((a, b)\), but on \((a, b)\), \( g = f \). Conclude \( f \) is bounded on \((a, b)\). □

Alternate Solution: Deny! Suppose \( f \) is not bounded. Then there exists\(^3\) a sequence \( \{x_n\} \) in \((a, b)\) such that \(|f(x_n)| \geq n\). Since \( \{x_n\} \) is a bounded sequence, by the Bolzano-Weierstrass Theorem, there is a convergent subsequence \( \{x_{p_n}\} \) of \( \{x_n\} \).

Since \( \{x_{p_n}\} \) is convergent, it also is Cauchy. By a proposition proved in class, \( \{x_{p_n}\} \) Cauchy and \( f \) uniformly continuous implies \( \{f(x_{p_n})\} \) is a Cauchy sequence. Finally, \( \{f(x_{p_n})\} \) Cauchy implies \( \{f(x_{p_n})\} \) is a bounded sequence.

We have shown that \( \{f(x_{p_n})\} \) is a bounded sequence, but from the original construction, we have \(|f(x_{p_n})| \geq p_n\), which means \( \{f(x_{p_n})\} \) is not a bounded sequence. Contradiction! □

Problem 13. Prove that if \( f \) has one-sided limits at \( a \) and \( b \), then \( f \) is uniformly continuous on \((a, b)\).

Proof: Extend the definition of \( f \) to include the endpoints: Define
\[
g(x) = \begin{cases} 
  f(a^-) & x = a \\
  f(x) & x \in (a, b) \\
  f(b^+) & x = b 
\end{cases}
\]

It is easy to argue that \( g \) is continuous on \([a, b]\), by Theorem 3.28, \( g \) is uniformly continuous on \([a, b]\).

To show \( f \) is u.c. on \((a, b)\), let \( \epsilon > 0 \). Since \( g \) is u.c. on \([a, b]\), there exists a \( \delta > 0 \) such that
\[
x, y \in [a, b] \land |x - y| < \delta \implies |g(x) - g(y)| < \epsilon
\]
(1)

Now, let \( x, y \in (a, b) \) such that \(|x - y| < \delta\), from (1), we know \(|g(x) - g(y)| < \epsilon\). But \( g(x) = f(x) \) and \( g(y) = f(y) \), for \( x, y \in (a, b) \), it follows from (1) then that
\[
|f(x) - f(y)| < \epsilon, \quad \text{whenever} \ x, y \in (a, b) \\text{ s.t. } |x - y| < \delta \quad □
\]

\(^3\)Standard techniques are used here.