§3.2, pages 97–99.

**Problem 4.** Let \( f : [a, b] \to \mathbb{R} \) be continuous at \( c \in [a, b] \) and suppose \( f(c) > 0 \). Prove that there is a number \( m > 0 \) and an interval \( [u, v] \subseteq [a, b] \) such that \( c \in [u, v] \) and \( f(x) \geq m, \forall x \in [u, v] \)

*Proof*: This is just a simple application of the definition of continuity at \( c \). Let \( \epsilon = f(c)/2 > 0 \), there since \( f \) is continuous at \( c \), there is a number \( \delta_1 > 0 \) such that
\[
x \in [a, b] \text{ and } |x - c| < \delta_1 \implies |f(x) - f(c)| < \epsilon
\]
Define \( \delta = \delta_1/2 \), then
\[
x \in [a, b] \text{ and } |x - c| \leq \delta \implies |f(x) - f(c)| < \epsilon \quad (1)
\]
Define \( m = f(c)/2 > 0 \), then
\[
|f(x) - f(c)| < \epsilon \implies f(x) \geq f(c)/2 = m \quad (2)
\]
To finish off this argument, we need to choose \( u \) and \( v \).

After much thought and taking into account the possibility that the endpoints of the \( \delta \) interval around \( c \) could extend beyond the endpoints of the interval \( [a, b] \), and that \( c \) could be either \( a \) or \( b \), we choose \( u = \max\{c - \delta, a\} \) and \( v = \min\{c + \delta, b\} \).

Thus,
\[
x \in [u, v] \implies x \in [a, b] \text{ and } |x - c| \leq \delta
\]
\[
\implies |f(x) - f(c)| < \epsilon \quad \text{from (1)}
\]
\[
\implies f(x) \geq f(c)/2 = m \quad \text{from (2)}
\]
\[
\implies f(x) \geq m
\]
This completes the proof, with all of its technical requirements. \(\square\)

**Problem 7.** (*Hint*: First argue that for each irrational number, \( x \), there exists a sequence \( \{r_n\} \) of rational numbers converging to \( x \).)

*Proof*: Suppose \( f \) is continuous on \( [a, b] \) and that \( f(x) = 0 \) for all \( x \in \mathbb{Q} \cap [a, b] \). We need to prove that \( f(x) = 0 \) for all \( x \in [a, b] \).

Let \( x \in [a, b] \). As was proven in the solutions to Assignment #7, there exists a sequence \( \{r_n\} \) of rational numbers (in \([a, b]\)) such that \( r_n \to x \). Since \( f \) is continuous at \( x \) we have
\[
\lim_{n \to \infty} f(r_n) = f(x) \quad (1)
\]
But, for each \( n \in \mathbb{N} \), \( r_n \in \mathbb{Q} \) so \( f(r_n) = 0 \), from the given properties of \( f \). But \( f(r_n) = 0, \forall n \in \mathbb{N} \) implies
\[
\lim_{n \to \infty} f(r_n) = 0
\]
But the limit of a sequence is unique (Theorem 2.4), combining the result (2) with
(1) yields \( f(x) = 0 \), which is what we wanted to prove, and completes the proof. \( \square \)

**Alternate Solution:** Let \( c \in [a, b] \) and let \( \epsilon > 0 \) be any number, then there is a \( \delta > 0 \)
such that \( x \in (c - \delta, c + \delta) \) implies \( |f(x) - f(c)| < \epsilon \). By Theorem 1.18, we can
choose a rational number \( r \in (c - \delta, c + \delta) \), then \( |f(c)| = |f(r) - f(c)| < \epsilon \).

We have shown that if \( c \in \mathbb{R} \), then for every \( \epsilon > 0 \), \( |f(c)| < \epsilon \). We conclude that
\( f(c) = 0 \). \( \square \)

**Problem 11(a).** Suppose \( f \) is continuous on \( I \), prove \( |f| \) is continuous on \( I \).

**Proof:** Let \( c \) be any element of \( I \), prove \( |f| \) is continuous on \( c \). Let \( \epsilon > 0 \) be given, \( f \)
is continuous at \( c \) implies there is a \( \delta > 0 \) such that
\[
x \in I \text{ and } |x - c| < \delta \implies |f(x) - f(c)| < \epsilon
\]
By the infamous **Reverse Triangle Inequality**, \( | |f(x)| - |f(c)| | \leq |f(x) - f(c)| \), so we have
\[
x \in I \text{ and } |x - c| < \delta \implies | |f(x)| - |f(c)| | \leq |f(x) - f(c)| < \epsilon
\implies | |f(x)| - |f(c)| | < \epsilon
\]
That is all. Over and out. \( \square \)

**Alternate Solution:** It is clear that the function \( g(x) = |x| \) is continuous on \( \mathbb{R} \), and
that \( f \) is continuous on \( I \). Since \( f(I) \subseteq \mathbb{R} \), it follows from Theorem 3.13 that \( g \circ f \)
is continuous on \( I \), but \( (g \circ f)(x) = |f(x)| \), so we can say that \( g \circ f = |f| \) is continuous
on \( I \). \( \square \)

**Problem 12.** This one is simple.

**Solution:** Let \( f \) and \( g \) be continuous on \( I \). Then by Corollary 3.12, the functions
\( f + g \) and \( f - g \) are continuous on \( I \). From the previous problem, since \( f - g \) is
continuous on \( I \), so is \( |f - g| \). Finally, by Corollary 3.12 we have
\[
f \lor g = \frac{(f + g) + |f - g|}{2}
\]
is continuous on \( I \) and
\[
f \land g = \frac{(f + g) - |f - g|}{2}
\]
is continuous on \( I \). \( \square \)

**Problem 28.** One such example is \( f(x) = \sin(1/(1 - x^2)) \) is one such example. \( \square \)