§3.1, pages 90–92.

Problem 4(b)(f).
(b) $\lim_{x \to 1} (x^2 + x - 1) = 1$.

Solution: Write the difference of the polynomial and its limit as a polynomial around $x = 1$:

$$(x^2 + x - 1) - 1 = x^2 + x - 2 = ((x - 1) + 1)^2 + (x - 1) - 2 = (x - 1)^2 + 3(x - 1)$$

Let $\epsilon > 0$ be given, we can assume $\delta$, yet to be determined satisfies $0 < \delta < 1$. Thus, assuming that $0 < |x - 1| < \delta$ we have

$$|(x^2 + x - 1) - 1| = |(x - 1)^2 + 3(x - 1)| < \delta^2 + 3\delta \leq \delta + 3\delta = 4\delta$$

From this calculation, we see that if we take $\delta = \min\{1, \epsilon/4\}$ we would have

$$0 < |x - 1| < \delta \implies |(x^2 + x - 1) - 1| < 4\delta \leq \epsilon$$

and that is what we wanted to prove. □

(f) Show $\lim_{x \to 1} \frac{4}{3x + 2} = \frac{4}{5}$.

Solution: We have

$$\left| \frac{4}{3x + 2} - \frac{4}{5} \right| = \frac{12}{5} \frac{|x - 1|}{|3x + 4|} \quad (1)$$

For a given $\epsilon > 0$, we can always assume $0 < \delta < 1$. In this case, for $0 < |x - 1| < \delta$, we see that $0 < x < 2$, hence $2 < 3x + 2 < 8$. Continuing the calculations from (1), we get,

$$\left| \frac{4}{3x + 2} - \frac{4}{5} \right| = \frac{12}{5} \frac{|x - 1|}{|3x + 4|} \leq \frac{12}{5} \cdot 2 \frac{|x - 1|}{|x - 1|} = \frac{6}{5} |x - 1| < \frac{6}{5}\delta \quad (2)$$

Now it is easy to see that we simply take $\delta = \min\{1, 5\epsilon/6\}$ and we will have from (2)

$$0 < |x - 1| < \delta \implies \left| \frac{4}{3x + 2} - \frac{4}{5} \right| < \frac{6}{5}\delta \leq \epsilon$$

which is what we wanted to prove. □

Problem 11. Let $I$ be an open interval that contains the point $c$, and let $f$ be a function that is defined on $I/\{c\}$.

(a) Suppose $\lim_{x \to c} f(x)$ exists. Prove $\lim_{x \to c} |f(x)|$ exists.

Proof: Let $\lim_{x \to c} f(x) = L$, from the Reverse Triangle Inequality, we have

$$| |f(x)| - |L| | \leq |f(x) - L|$$
The result follows from this inequality. □

(b) Suppose \( \lim_{x \to c} |f(x)| \) exists, show by example that \( \lim_{x \to c} f(x) \) need not exist.

**Solution:** Define the function

\[
f(x) = \begin{cases} 
1 & \text{if } x \text{ is rational} \\
-1 & \text{if } x \text{ is irrational}
\end{cases}
\]

Then \( |f(x)| = 1, \forall x \in \mathbb{R} \). But \( \lim_{x \to c} f(x) \) does not exist for any \( c \in \mathbb{R} \). (See Problem 13 for a proof.) □

(c) Show \( \lim_{x \to 0} |f(x)| = 0 \implies \lim_{x \to 0} f(x) = 0 \).

**Proof:** Trivial. We’ve done this type thing before. □

**Problem 13.** Define \( f(x) = 0 \) for \( x \) irrational and \( f(x) = 1 \) for \( x \) rational. Prove that \( f \) does not have a limit at any point.

Before we begin the proof, let’s enter a Lemma into the class record:

**Lemma:** For any \( x \in \mathbb{R} \), there exists a sequence \( \{r_n\} \) of rational numbers and a sequence \( \{w_n\} \) of irrational numbers s.t. \( \{r_n\} \) converges to \( x \) and \( \{w_n\} \) converges to \( x \).

**Proof:** Let \( \{\epsilon_n\} \) be any sequence of positive numbers converging to zero. For each \( n \in \mathbb{N} \), by Theorem 1.18, there exists a rational and an irrational number in the interval \((x - \epsilon_n, x + \epsilon_n)\); call these two numbers, whose existence is guaranteed \( r_n \) and \( w_n \). Thus, for each \( n \in \mathbb{N} \) we have,

\[
|r_n - x| < \epsilon_n \quad \text{and} \quad |w_n - x| < \epsilon_n
\]

It follows from the Squeeze Theorem for Sequences that \( \{r_n\} \) converges to \( x \) and \( \{w_n\} \) converges to \( x \). □

Now, let’s prove Problem 13.

**Proof:** Let \( x \) be any real number, by the Lemma, there exists a sequence \( \{r_n\} \) of rational numbers and a sequence \( \{w_n\} \) of irrational numbers s.t. \( \{r_n\} \) converges to \( x \) and \( \{w_n\} \) converges to \( x \). Then

\[
\lim_{n \to \infty} f(r_n) = 1 \quad \text{since } r_n \text{ is rational} \\
\lim_{n \to \infty} f(w_n) = 0 \quad \text{since } w_n \text{ is irrational}
\]

It follows from Theorem 3.2(b) that \( f \) does not have a limit at \( x \). Since \( x \) was an arbitrary real number, we have \( f \) does not have a limit at any point. □
Problem 34(a)(c). Supply some definitions.

(a) We say that \( \lim_{x \to c^+} f(x) = -\infty \) if for each \( M < 0 \), there exists a \( \delta > 0 \) such that if \( x \in (c, c + \delta) \cap I \), then \( f(x) < M \).

(c) We say that \( \lim_{x \to \infty} f(x) = \infty \) if for each \( M > 0 \), there exists a \( b > 0 \) such that if \( x > b \) then \( f(x) > M \).


(a) Prove \( \lim_{x \to 2^+} \frac{1}{x^2 - 2x} = \infty \).

**Proof:** Let \( M > 0 \) be given and let \( 0 < \delta < 1 \) be any number. Suppose \( x \in (2, 2 + \delta) \). Then \( 2 < x < 3 \) (since \( \delta < 1 \)).

\[
\frac{1}{x^2 - 2x} = \frac{1}{x(x - 2)} \geq \frac{1}{3(x - 2)} \tag{1}
\]

It is clear that if we take \( \delta = \min\{1, 1/(3M)\} \), and \( 0 < |x - 1| < \delta \), from (1) we have

\[
\frac{1}{x^2 - 2x} \geq \frac{1}{3(x - 2)} \geq \frac{1}{3} 3M = M
\]

which is what we wanted to prove. \( \square \)

(b) Prove \( \lim_{x \to \infty} \frac{x^2 - 1}{5x - 2x^2} = -\frac{1}{2} \).

**Solution:** Consider

\[
\left| \frac{x^2 - 1}{5x - 2x^2} + \frac{1}{2} \right| = \frac{1}{2} \frac{|5x - 2|}{x(2x - 5)} \tag{1}
\]

Let \( \epsilon > 0 \) be given, and let \( b > 3 \). For \( x > b > 3 \) we have from (1)

\[
\left| \frac{x^2 - 1}{5x - 2x^2} + \frac{1}{2} \right| = \frac{1}{2} \frac{|5x - 2|}{x(2x - 5)} \leq \frac{5x - 2}{x(2x - 5)} \leq \frac{5x}{x(2x - 5)} \leq \frac{5}{2x - 5} \leq \frac{5}{2x - 5} \tag{2}
\]

Note that when \( x > 3 \), we have \( 5x > 5x - 2 > 0 \) and \( 2x - 5 > 0 \). We also used the fact that \( 1/2 \leq 1 \).

Now we proudly declare \( b \) to be any number such that \( b > \max\{3, (5/2)(1 + 1/\epsilon)\} \).

From (2), we now have

\[
x > b \implies \left| \frac{x^2 - 1}{5x - 2x^2} + \frac{1}{2} \right| \leq \frac{5}{2x - 5} < \epsilon
\]

which is what we wanted to prove. \( \square \)